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Dynamics of inner functions revisited

by

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Abstract. We study the circle restrictions of inner functions on the unit disc showing that the local invertibility of a restriction is independent of its singularity set and proving a local characterization of analytic conditional expectations.

We establish central limit properties for some stochastic processes driven by probability preserving restrictions via spectral analysis of their perturbed transfer operators.

1. Introduction

1.1. Inner functions and their restrictions. An *inner function* on the *unit disc* $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is an analytic endomorphism $\phi : \mathbb{D} \circlearrowleft$ such that for Lebesgue almost every $\xi \in \partial\mathbb{D}$,

$$\phi(r\xi) \xrightarrow[r\to 1-]{} \phi(\xi) \in \partial\mathbb{D}.$$

The *restriction* of ϕ to $\partial\mathbb{D}$ is defined λ-a.s. (where λ is Lebesgue measure on $\partial\mathbb{D}$) and is a *nonsingular transformation* of $(\partial\mathbb{D}, \lambda)$ in the sense that λ and $\lambda \circ \phi^{-1}$ have the same null sets.

This follows from Nordgren's theorem (in §2.3), which also shows the connection between the ergodic theory of an inner function restriction on $(\partial\mathbb{D}, \lambda)$ and the dynamics of the action of the inner function on $\mathbb{D}$.

For this, and more discussion of the ergodic theory of restrictions, see e.g. [Aar97, Ch. 6] and references therein.

1.2. Overview of the paper. This paper deals with the structure and properties of inner functions, the spectral theory of their transfer operators and the central limit theory of stochastic processes driven by their restrictions.

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The rest of this subsection is devoted to a description of the main results.

Local invertibility vs. singularity set. In §2.2 we consider *local invertibility* of general nonsingular maps showing i.a. that a nonsingular transformation of a standard, nonatomic probability space is locally invertible iff it is *forward nonsingular* (Theorem 2.2).

This enables an elementary proof of a multiplicity result of Aleksandrov (Proposition 2.7): an inner function has a locally invertible restriction iff it admits angular derivatives a.e. on $\partial\mathbb{D}$.

The *singularity set* of an inner function (1) (see §2.6) was studied in [Sei34]. Evidently, if an inner function has a Lebesgue-null singularity set, then its restriction, being analytic a.e. on $\mathbb{T}$, is locally invertible. However (by Proposition 2.8), any closed set of $\partial\mathbb{D}$ appears as the singularity set of an inner function whose restriction is locally invertible.

Spectrum of the transfer operator and central limits. In §3 & §4, we restrict attention to non-Möbius inner functions $\phi : \mathbb{D} \circlearrowleft$ preserving an absolutely continuous probability, showing in §3 that their transfer operators have spectral gaps (2) on weighted Hilbert spaces (see §3.3).

In §4 we consider central limit properties of stochastic processes $(\psi \circ \tau^n : n \geq 1)$ $(\psi : \mathbb{T} \to \mathbb{R}^d)$ driven by such restrictions $\tau = \tau(\phi)$. In particular, if $\phi : \mathbb{D} \circlearrowleft$ has a singularity on $\partial\mathbb{D}$ and $\psi : \mathbb{T} \to \mathbb{R}$ is real analytic and not constant, then $(\psi \circ \tau^n : n \geq 1)$ satisfies the conditional central limit theorem (as in (32)). These results rely on smooth perturbations of the quasicompact transfer operators under consideration.

The central limit theorem for measure preserving inner function restrictions (as in (22)) is established in [IU23] for e.g. nonconstant Hölder continuous functions. See also [NSiG22] for different forms of central limit theorem for measure preserving inner function restrictions.

Analytic conditional expectations. Aleksandrov [Ale86] proved that conditional expectation with respect to a sub-σ-algebra $\mathcal{C} \subset \mathcal{B}(\mathbb{T})$ is *analytic* in the sense that the conditional expectation projection commutes with the *Riesz projection* (orthogonal projection $L^2 \to H_0^2 := \{f \in H^2 : \hat{f}(0) = 0\}$) if and only if $\mathcal{C} = \tau^{-1}\mathcal{B}(\mathbb{T})$, where $\tau = \tau(\phi)$ with ϕ inner, $\phi(0) = 0$.

In §5 we prove a local version of this (Theorem 5.2).

2. Structure and basic properties of inner functions

2.1. Nonsingular maps and transformations. A *nonsingular map* $\pi : (X, m) \to (Y, \mu)$ between the nonatomic, Polish probability spaces (X, m)

(1) The (closed) set of points of $\partial\mathbb{D}$ where it is not analytic.

(2) That is, are Doeblin–Fortet operators (see Section 3.1).

and (Y, μ) is a measurable map $\pi : X \to Y$ satisfying $m \circ \pi^{-1} \sim \mu$. It is *probability preserving* if $m \circ T^{-1} = \mu$.

A *nonsingular* [probability preserving] *transformation* (X, m, T) is a nonsingular [probability preserving] self-map $T : (X, m) \to (X, m)$.

The *transfer operator* (also called "adjoint" or "predual") of the nonsingular map $\pi : (X, m) \to (Y, \mu)$ is $\widehat{\pi} : L^1(m) \to L^1(\mu)$ defined by

$$\int_A \widehat{\pi} f \, d\mu = \int_{\pi^{-1}A} f \, dm, \qquad A \in \mathcal{B}(X).$$

The following is a standard application of the disintegration theorem ([DM78, Ch. III]; see also [Aar97, Ch. 1]):

PROPOSITION 2.1 (Preimage measures). *Let $\pi : (X, m) \to (Y, \mu)$ be a nonsingular map. Then there exist $Y_0 \in \mathcal{B}(X)$ with $\mu(Y_0) = 1$ and $\nu = \nu^{(\pi)} : Y_0 \to \mathcal{M}(X)$ such that*

(1) $\quad \widehat{\pi} 1_A(y) = \nu_y(A)$ *for* $y \in Y_0$, $A \in \mathcal{B}(X)$, *and* $\nu_y(X) = \dfrac{dm \circ \pi^{-1}}{d\mu}(x)$;

(2) $\quad \nu_y(X \setminus \pi^{-1}\{y\}) = 0 \quad$ *and* $\quad \nu_x \perp m$ *for* μ*-a.e.* $y \in Y_0$.

The function $x \mapsto \nu_x$ is known as the *transition kernel* of $\widehat{\pi}$ and the ν_x are known as *preimage measures* or *fiber measures*.

2.2. Local invertibility and forward nonsingularity. We will say that a nonsingular map $\pi : (X, m) \to (Y, \mu)$ is

- *almost countable-to-one* if there exists $Y_0 \in \mathcal{B}(Y)$ with $\mu(Y \setminus Y_0) = 0$ such that $\pi^{-1}\{y\} \cap Y_0$ is at most countable for all $y \in Y_0$;
- *locally invertible* if there exists an at most countable partition $\alpha \subset \mathcal{B}(Y)$ such that $\pi : a \to \pi a$ is invertible and nonsingular for all $a \in \alpha$;
- *forward nonsingular* if there exists $X_0 \in \mathcal{B}(X)$ and $m(X \setminus X_0) = 0$ such that
$$[A \in \mathcal{B}(X_0),\ m(A) = 0] \implies \mu(\pi A) = 0.$$

As shown in [Rok61] (see also [Nad81] and [Aar97, Ch. 1]), for the nonsingular map $\pi : (X, m) \to (Y, \mu)$ the "almost countable-to-one" and "locally invertible" conditions are both equivalent to the pure atomicity of the transition kernel (i.e. almost every preimage measure is purely atomic).

THEOREM 2.2. *A nonsingular map $\pi : (X, m) \to (Y, \mu)$ is forward nonsingular iff it is locally invertible.*

Proof. It is standard that local invertibility implies forward nonsingularity. We will show that forward nonsingularity implies pure atomicity of the transition kernel.

Let $y \mapsto \nu_y$ $(Y_0 \to \mathcal{M}(X))$ be the transition kernel of $\widehat{\pi}$.

We claim first that the function $x \mapsto \nu_{\pi x}(\{x\})$ is measurable.

To see this let d be a Polish metric on X and let α_n be a sequence of partitions of X such that $\alpha_{n+1} \succ \alpha_n$, $\sup_{a\in\alpha_n} \mathtt{diam}\, a \xrightarrow[n\to\infty]{} 0$. Thus $\alpha_n(y) \downarrow \{y\}$ for all $y \in X$ (where $y \in \alpha_n(y) \in \alpha_n$) and the function $y \mapsto \nu_{\pi y}(\{y\}) = \lim_{n\to\infty} \nu_{\pi y}(\alpha_n(y))$ is measurable.

It follows that $W := \{x \in X : \nu_{\pi x}(\{x\}) = 0\} \in \mathcal{B}(X)$.

If the transition kernel is not purely atomic, then $m(W) > 0$.

Next, πW is analytic, whence universally measurable, and since $\pi^{-1}\pi W \supseteq W$, we have $m(\pi^{-1}\pi W) \geq m(W) > 0$, whence (by nonsingularity of π) $\mu(\pi W) > 0$.

For $V \in \mathcal{B}(W)$,

$$\begin{aligned} m(V) &= m(V \cap W) = m(V \cap \pi^{-1}\pi W) \qquad (\because\ \pi^{-1}\pi W \supset W) \\ &= \int_{\pi W} \widehat{\pi} 1_V \, d\mu = \int_{\pi W} \nu_x(V)\, d\mu(x). \end{aligned}$$

By the analytic section theorem [Lus30] (see also [Jan41], [vN49], [Sri98, Thm. 5.5.2], [Coh80, §8.5]), and Luzin's continuity theorem, there exists $A \subset \pi W$ compact with $\mu(A) > 0$ and $\zeta : A \to B := \zeta A \subset W$ continuous such that $\pi \circ \zeta = \mathtt{Id}$. It follows that

$$B \cap \pi^{-1}\{x\} = \begin{cases} \emptyset, & x \notin A, \\ \{\zeta(x)\} & x \in A, \end{cases}$$

whence

$$\nu_x(B) = 1_A(x)\nu_x(\{\zeta(x)\}) = 0 \qquad (\because \pi(x) \in W \ \&\ x = \pi(\zeta(x))).$$

Thus

$$m(B) = \int_A \nu_x(\{\zeta(x)\})\, d\mu(x) = 0,$$

whereas $\pi B = \pi\zeta A = A$, $\mu(\pi B) = \mu(A) > 0$ and forward nonsingularity fails. ∎

2.3. Structure of inner function restrictions. It will be convenient to identify $\partial\mathbb{D}$ with $\mathbb{T} := \mathbb{R}/\mathbb{Z} \cong [0, 1)$ via

$$x \in [0, 1) \leftrightarrow \chi(t) := e^{2\pi i x} \in \partial\mathbb{D}.$$

Indeed, $\chi : (\mathbb{T}, m) \to (\partial\mathbb{D}, \lambda)$ is an isomorphism of measure spaces where m is Lebesgue measure on $[0, 1)$.

Let $\phi : \mathbb{D} \circlearrowleft$ be inner with restriction $\phi : \partial\mathbb{D} \circlearrowleft$. Define $\tau = \tau(\phi) : \mathbb{T} \circlearrowleft$ m-a.e. by $\tau(t) := \chi^{-1}(\varphi(\chi(t)))$.

The following shows that $(\mathbb{T}, m, \tau)$ is a nonsingular transformation, whence so is the isomorphic $(\partial\mathbb{D}, \lambda, \varphi)$.

NORDGREN'S THEOREM ([Nor68]). *For $\phi : \mathbb{D} \circlearrowleft$ inner,*

$$\pi_z \circ \tau^{-1} = \pi_{\phi(z)}, \quad \textit{or equivalently} \quad \widehat{\tau} p_z = p_{\phi(z)}, \tag{3}$$

where π_z $[p_z]$ is the Poisson measure [density] at z *defined by*

$$d\pi_z(x) := p_z(x)dx, \qquad p_z(x) := \mathsf{Re}\, \frac{\chi(x) + z}{\chi(x) - z}. \tag{4}$$

REMARK. Equation (3) is also known as *Boole's formula* because a version for rational inner functions of the upper half-plane appears in [Boo57, p. 787] (see also [Gla77, §8]). For a converse to Nordgren's theorem, see [Let77].

2.4. Invariant probabilities for inner function restrictions

DENJOY–WOLFF THEOREM ([Den26, Wol26]). *Suppose that $\phi : \mathbb{D} \circlearrowleft$ is analytic, nonconstant and non-Möbius. Then there is a (unique) point $\mathfrak{d} = \mathfrak{d}(\phi) \in \overline{\mathbb{D}}$ such that*

$$\frac{1 - |\phi(z)|^2}{|1 - \overline{\mathfrak{d}}\phi(z)|^2} \geq \frac{1 - |z|^2}{|1 - \overline{\mathfrak{d}} z|^2} \qquad \forall z \in \mathbb{D},$$

and

$$\phi^n(z) \to \mathfrak{d} \qquad \textit{as } n \to \infty\ \forall z \in \mathbb{D}.$$

The point $\mathfrak{d}(\phi)$ is called the *Denjoy–Wolff point* of ϕ.

COROLLARY 2.3 (see e.g. [Aar78, Neu78, DM91]). *The restriction τ of an inner function ϕ has an absolutely continuous, invariant probability iff the Denjoy–Wolff point of ϕ belongs to $\mathbb{D}$, and in this case $(\mathbb{T}, \pi_{\mathfrak{d}(\phi)}, T)$ either is conjugate to a circle rotation or is an exact probability preserving transformation.*

2.4.1. *Clark measures.* As in [Sak07], the *Clark measure* of an analytic endomorphism $\phi : \mathbb{D} \circlearrowleft$ at $\xi \in \partial\mathbb{D}$ is the representing measure $\mu_\xi = \mu_\xi^{(\phi)} \in \mathcal{M}(\mathbb{T})$ of the positive harmonic function $z \mapsto \mathsf{Re}\, \frac{\xi + \phi(z)}{\xi - \phi(z)}$ satisfying

$$\mathsf{Re}\, \frac{\xi + \phi(z)}{\xi - \phi(z)} = \int_{\mathbb{T}} p_z\, d\mu_\xi. \tag{5}$$

It follows that $\xi \mapsto \mu_\xi$ is weak* continuous $\partial\mathbb{D} \to \mathcal{M}(\mathbb{T})$.

Now let $\phi : \mathbb{D} \circlearrowleft$ be inner. Then $\mathsf{Re}\, \frac{\xi + \phi(r\chi)}{\xi - \phi(r\chi)} \xrightarrow[r \to 1-]{} 0$ a.s., whereas by Fatou's theorem,

$$\int_{\mathbb{T}} p_{r\chi} d\mu_\xi \xrightarrow[r \to 1-]{} \frac{d\mu_\xi}{dm} \text{ a.s.} \quad \text{and} \quad \mu_\xi \perp m\ \forall \xi \in \partial\mathbb{D}.$$

Moreover, since $p_z(t) = \mathsf{Re}\, \frac{\chi(t) + z}{\chi(t) - z}$, it follows from (3) that $\nu_t^{(\tau)} = \mu_{\chi(t)}$ where $\nu^{(\tau)} : \mathbb{T} \to \mathcal{M}(\mathbb{T})$ are the preimage measures of the restriction $\tau = \tau(\phi)$. See also [Sak07, §2].

2.5. Factorization of inner functions

2.5.1. *Blaschke products.* Suppose that $Z \subset \mathbb{D}$ is countable and that $\mathfrak{m} : Z \to \mathbb{N}$ is such that $\sum_{a \in Z} \mathfrak{m}(a)(1 - |a|) < \infty$.

The *Blaschke product* with zero set Z and *multiplicity function* $\mathfrak{m}$ is $B = B_{Z,\mathfrak{m}} : \mathbb{D} \to \mathbb{C}$ defined by

$$B(z) := \prod_{\alpha \in Z} b_\alpha(z)^{\mathfrak{m}(\alpha)} \quad \text{with} \quad b_\alpha(z) := c_\alpha \frac{z - \alpha}{1 - \overline{\alpha} z},\ c_\alpha = \frac{-\overline{\alpha}}{\alpha}\ (\alpha \neq 0),\ c_0 = 1.$$

The product converges locally uniformly on $\mathbb{D}$ because

$$|1 - b_\alpha(z)^{\mathfrak{m}(\alpha)}| \le \frac{1 + |z|}{1 - |z|} \cdot \mathfrak{m}(\alpha)(1 - |\alpha|).$$

It can be shown that B is inner, $\{a \in \mathbb{D} : B(a) = 0\} = Z$ and for $a \in Z$, $B(z)/b_a(z)^{\mathfrak{m}(a)} =: H(z)$ is bounded and analytic with $H(a) \neq 0$.

2.5.2. *Singular inner functions.* An inner function $S : \mathbb{D} \circlearrowleft$ is *singular* (i.e. without zeros) iff

$$\log S(z) = -\int_{\mathbb{T}} \frac{\chi(t) + z}{\chi(t) - z}\, d\sigma(t),$$

where $\sigma \in \mathfrak{M}(\mathbb{T})$ with $\sigma \perp m$.

In this situation, we write $S = S_\sigma$ (&/or $\sigma = \sigma_S$).

Factorization Theorem ([Smi29]; see also [Rud74, Thm. 17.15]). *Let $\phi : \mathbb{D} \circlearrowleft$ be an inner function, let $\mathcal{Z}(\phi) := \{z \in \mathbb{D} : \phi(z) = 0\}$ and, for $a \in \mathcal{Z}(\phi)$, let*

$$\mathfrak{m}(a) := \max\{k \ge 0 : \phi/b_a^k \text{ is bounded on } \mathbb{D}\}.$$

Then $\sum_{\alpha \in \mathcal{Z}(\phi)} \mathfrak{m}(a)(1 - |\alpha|) < \infty$ and $\phi = \lambda B_{\mathcal{Z}(\phi),\mathfrak{m}} \cdot S$, where $\lambda \in \mathbb{S}^1$ and S is a singular inner function.

2.6. Regular points and singularities.

A *regular point* of the inner function $\phi : \mathbb{D} \circlearrowleft$ is a point $z \in \partial\mathbb{D}$ such that there exists $U \subset \mathbb{C}$ open with $z \in U$ and an analytic function $F : U \to \mathbb{C}$ such that $F|_{U \cap \mathbb{D}} \equiv \phi|_{U \cap \mathbb{D}}$.

A nonregular point in $\partial\mathbb{D}$ is called a *singularity.*

The *regularity set* of ϕ is $\mathfrak{r}_\phi := \{\text{regular points of } \phi\}$ and the *singularity set* of ϕ is $\mathfrak{s}_\phi := \partial\mathbb{D} \setminus \mathfrak{r}_\phi$.

- If ν_x is any Clark measure for the inner function ϕ (as in §2.4.1) then $\chi^{-1}\mathfrak{s}_\phi = (\mathtt{spt}\,\nu_x)'$.
- If $S = S_\sigma$ is a singular inner function then $\chi^{-1}\mathfrak{s}_S = \mathtt{spt}\,\sigma$.

2.6.1. *Derivative of a restriction at a regular point.* Let $\phi : \mathbb{D} \circlearrowleft$ with restriction $\tau = \tau(\phi)$. Then τ is differentiable at each $\theta \in \chi^{-1}\mathfrak{r}_\phi$ with

$$\tau'(\theta) = \partial\phi(\chi(\theta)) = |\phi'(\chi(\theta))| \quad \text{with} \quad \partial\phi(z) := \frac{z\phi'(z)}{\phi(z)} = z(\log\phi)'(z). \tag{6}$$

If $\phi = \lambda B_{\mathcal{Z}(\phi),\mathfrak{m}} S_\sigma$, then

$$\mathfrak{s}_\phi = \mathcal{Z}(\phi)' \cup \mathtt{spt}\,\sigma$$

and, as in [Mar89],

$$\begin{aligned} (7)\quad \tau'(\theta) &= \sum_{\alpha\in\mathcal{Z}(\phi)} \mathfrak{m}(\alpha) p_\alpha(\theta) + \frac{1}{2}\int_{\mathbb{T}} \frac{d\sigma(t)}{\sin^2(\pi(\theta - t))} \\ &\geq \sum_{\alpha\in\mathcal{Z}(\phi)} \frac{1-|\alpha|}{1+|\alpha|} + \frac{\sigma(\mathbb{T})}{2} =: \eta \geq 0 \quad \forall\theta\in\mathbb{T} \text{ such that } \chi(\theta)\in\mathfrak{r}_\phi, \end{aligned}$$

where $p_\alpha(\theta) := \mathtt{Re}\,\frac{\alpha+\chi(t)}{\alpha-\chi(t)}$ and $\sum_{\alpha\in\emptyset} := 0$. Since either $\mathcal{Z}(\phi)\neq\emptyset$ or $\sigma\neq 0$ (or both), we have $\eta > 0$.

2.6.2. *Arc maps.* An *arc map* is a triple $(\mathbb{T}, T, \alpha)$, where α is a finite or countable partition mod m of $\mathbb{T}$ into *open arcs* (open, connected subsets $A \subsetneq \mathbb{T}$) (3) and $T : \mathbb{T}\to\mathbb{T}$ is a map such that

- for each $A\in\alpha$, $T : A \to T(A)$ is a bi-absolutely continuous homeomorphism,
- $\sigma(\bigcup_{n\geq 0} T^{-n}\alpha) = \mathcal{B}(\mathbb{T})$.

It is called *piecewise* C^k $(k\geq 1)$ [*analytic*] if each $T : A\to TA$ is a C^k-diffeomorphism [bi-analytic].

Proposition 2.4 (Arc map restrictions). *Let $\phi : \mathbb{D} \circlearrowleft$ be inner with $m(\mathfrak{s}_\phi) = 0$ and suppose the Denjoy–Wolff point of ϕ belongs to $\mathbb{D}$. Then:*

(i) *$\tau : \mathbb{T}\circlearrowleft$ defines a piecewise analytic, arc map $(\mathbb{T},\tau,\alpha)$ such that*

$$(8)\qquad \begin{aligned} &\tau'(x) \geq \eta > 0 \quad \forall x\in\chi^{-1}\mathfrak{r}_\phi, \\ &\exists d\geq 1 :\ |(\tau^d)'(x)| \geq \beta > 1 \quad \forall x \in \bigcap_{j=0}^{d-1} \tau^{-j}\chi^{-1}\mathfrak{r}_\phi. \end{aligned}$$

(ii) *If in addition $\#\mathcal{Z}(\phi), \#\mathfrak{s}_\phi < \infty$, then the partition α may be chosen to be* surjective*:*

$$(9)\qquad \tau(A) = \mathbb{T} \bmod m \quad \forall A\in\alpha.$$

Moreover,

$$(10)\qquad \sup_{x\in\chi^{-1}\mathfrak{r}_\phi} \varDelta\tau(x) < \infty \quad \textit{with} \quad \varDelta\tau(x) := \frac{|\tau''(x)|}{\tau'(x)^2}.$$

We will call piecewise onto arc maps satisfying (8)–(10) *Adler maps.*

Adler interval maps (Adler arc maps with surjective partitions into intervals) are considered in [Adl73].

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(3) Of the form $\chi^{-1}A = (a,b)$ with $0\leq a<b\leq 1$ or $[0,a)\cup(b,1]$ with $0<a\leq b<1$.

Any Adler arc map is conjugate by rotation to an Adler interval map.

Proof of Proposition 2.5.

Proof of (8). There is a Möbius transformation $\Psi : \mathbb{D} \circlearrowleft$ such that $\mathfrak{d}(g) = 0$, where $g = \Psi^{-1} \circ \phi \circ \Psi$. If the restrictions of g & Ψ are U & ψ respectively, then $\tau^n = \psi \circ U^n \circ \psi^{-1}$ for all $n \geq 1$.

Since $g(0) = 0$, by (7) we have

$$U'(\theta) = \partial g(\chi(\theta)) = 1 + \partial h(\chi(\theta)) \geq 1 + \delta_h =: \rho > 1.$$

For $n \geq 1$,

$$\begin{aligned} \tau^{n\prime} &= \psi' \circ U^n \circ \psi^{-1} \cdot U^{n\prime} \circ \psi^{-1} \cdot \psi^{-1\prime} \\ &\geq \rho^n \min_{\mathbb{T}} \psi' \cdot \min_{\mathbb{T}} \psi^{-1\prime} \\ &\geq B > 1 \quad \text{ for large enough } n \geq 1. \end{aligned}$$

Proof that $\sigma(\bigcup_{n\geq 0} T^{-n}\alpha) = \mathcal{B}(\mathbb{T})$. By (8) we see that $\alpha_n := \bigvee_{k=0}^{n-1} \tau^{-1}\alpha$ is also a mod 0 partition of $\mathbb{T}$ into open arcs satisfying

$$\max\{m(a) : a \in \alpha_n\} \leq (1/\eta)^d \cdot (1/\beta)^{n/d},$$

whence $\sigma(\bigcup_{n\geq 0} \tau^{-n}\alpha) \overset{m}{=} \mathcal{B}(\mathbb{T})$.

Proof of (9) *in case* $\mathfrak{s}_\phi \neq \emptyset$. We construct a mod 0 partition α of $\mathbb{T}$ into open arcs satisfying (9).

Since $\#\mathfrak{s}_\phi < \infty$, formula (7) now has the form

$$\mathfrak{T}'(\theta) = \sum_{a \in \mathcal{Z}(\phi)} \mathfrak{m}(a) p_a(\theta) + \frac{1}{2} \sum_{t \in \mathfrak{s}_\phi} \frac{\sigma(\{t\})}{\sin^2(\pi(\theta - t))}, \tag{11}$$

Suppose that $J \subset \mathbb{T}$ is an open arc and $f : J \to \mathbb{T}$ is continuously differentiable on J with $\min_J f' > 0$. Then f has a *lifting*: there is an interval $\widetilde{J} \subset \mathbb{R}$ such that $\mathscr{m}\widetilde{J} = J$, where $\mathscr{m} : \mathbb{R} \to \mathbb{T}$, $\mathscr{m}(x) = x \bmod 1$; and there exists $F : \widetilde{J} \to \mathbb{R}$ continuously differentiable such that

$$\mathscr{m}(F(x)) = f(\mathscr{m}(x)) \quad \text{ for } x \in \widetilde{J}.$$

In particular, $F'(x) = f'(\mathscr{m}(x))$.

Let $J \subset \chi^{-1}\mathfrak{r}_\phi$ be a maximal open arc (i.e. $\partial J \subset \chi^{-1}\mathfrak{s}_\phi$) and let $\mathfrak{T} : \widetilde{J} \to \mathbb{R}$ be the lifting of $\tau : J \to \mathbb{T}$. Write $\widetilde{J} = (a_-, a_+)$. Then by (11),

$$\mathfrak{T}'(\theta) \xrightarrow[\theta \to \{a_-, a_+\},\, \theta \in (a_-, a_+)]{} \infty,$$

whence

$$\mathfrak{T}(\theta) \xrightarrow[\theta \to a_\pm,\, \theta \in (a_-, a_+)]{} \pm\infty \tag{12}$$

and there is a countable mod 0 partition $\mathscr{p}_{\widetilde{J}}$ of $\widetilde{J}$ into open arcs such that for each $A \in \mathscr{p}_{\widetilde{J}}$, $\mathfrak{T}A$ is an interval of length 1.

It follows that $\alpha_J := \mathscr{m}\mathscr{p}_{\tilde{J}}$ is a mod 0 partition of J into open arcs such that $\tau A = \mathbb{T}$ mod 0 for all $A \in \alpha_J$.

Since $\#\mathfrak{s}_\phi < \infty$ we know that $\chi^{-1}\mathfrak{r}_\phi$ is a finite union of maximal open arcs as above and so there is a mod 0 partition α of $\chi^{-1}\mathfrak{r}_\phi$ into open arcs such that $\tau A = \mathbb{T}$ mod 0 for all $A \in \alpha$.

Proof of (10). Since $\#\mathcal{Z}(\phi)$ and $\mathfrak{s}_\phi$ are both finite, by (7), $\tau(\theta) = \mathfrak{T}(\theta)$ mod 1 with

$$\mathfrak{T}(\theta) = \mathfrak{b}(\theta) - \frac{1}{2\pi}\sum_{t\in\mathfrak{s}_\phi} \sigma(\{t\})\cot(\pi(\theta - t)) \tag{13}$$

where $\mathfrak{b} \equiv 0$ when $\mathcal{Z}(\phi) = \emptyset$; and when $1 \le \#\mathcal{Z}(\phi) < \infty$,

$$\mathfrak{b}(\theta) = \int_0^\theta \Big(\sum_{a\in\mathcal{Z}(\phi)} \mathfrak{m}(a)p_a(t)\Big)\,dt \bmod 1$$

defines an analytic endomorphism of $\mathbb{T}$.

Since either $\mathfrak{b} = 0 = \varDelta(\mathfrak{b})$ or $\mathfrak{b} : \mathbb{T} \circlearrowleft$ is analytic expanding, in which case $\varDelta(\mathfrak{b}) : \mathbb{T} \to \mathbb{R}$ is analytic, we have $\|\varDelta(\mathfrak{b})\|_\infty < \infty$.

In case $\mathfrak{s}_\phi \neq \emptyset$, by (11) we have

$$\tau'(\theta) = \mathfrak{b}'(\theta) + \frac{1}{2}\sum_{t\in\mathfrak{s}_\phi} \frac{\sigma(\{t\})}{\sin^2(\pi(\theta - t))} =: \mathfrak{b}'(\theta) + s'(\theta) \tag{14}$$

and

$$|\tau''(\theta)| \le |\mathfrak{b}''(\theta)| + |s''(\theta)| \le |\mathfrak{b}''(\theta)| + \pi\sum_{t\in\mathfrak{s}_\phi}\sigma(\{t\})\left|\frac{\cos(\pi(\theta - t))}{\sin^3(\pi(\theta-t))}\right|,$$

whence

$$\begin{aligned}
\varDelta\tau(\theta) &\le \|\varDelta\mathfrak{b}\|_\infty + \pi\frac{\sum_{t\in\mathfrak{s}_\phi}\sigma(\{t\})\big|\frac{\cos(\pi(\theta-t))}{\sin^3(\pi(\theta-t))}\big|}{s'(\theta)^2}\\
&\le \|\varDelta\mathfrak{b}\|_\infty + \pi\sum_{t\in\mathfrak{s}_\phi}\frac{\frac{\sigma(\{t\})|\cos(\pi(\theta-t))|}{|\sin^3(\pi(\theta-t))|}}{\big(\frac{\sigma(\{t\})}{\sin^2(\pi(\theta-t))}\big)^2}\\
&= \|\varDelta\mathfrak{b}\|_\infty + \frac{\pi}{2}\sum_{t\in\mathfrak{s}_\phi}\frac{|\sin(2\pi(\theta-t))|}{\sigma(\{t\})} \le \|\varDelta\mathfrak{b}\|_\infty + \frac{\pi}{2}\sum_{t\in\mathfrak{s}_\phi}\frac{1}{\sigma(\{t\})}\\
&=: M < \infty. \ \blacksquare
\end{aligned}$$

EXAMPLE 2.5. We exhibit an inner function $\phi : \mathbb{D} \circlearrowleft$ with $\phi(0) = 0$ and singularity set $\mathfrak{s}_\phi = \{\chi(a), \chi(b)\}$ with $0 \le a < b \le 1$ such that $\tau = \tau(\phi) : [a, b] \to [\tau(a+), \tau(b-)] \subset (0, 1)$ is a homeomorphism. Thus the restriction is not an Adler arc map. Indeed, such an inner function must be a Blaschke product since otherwise either $\chi(a)$ is a singularity of the singular factor of

ϕ and by (12), $\tau(a-)=0$, or $\chi(b)$ is and $\tau(b-)=1$. This would contradict $[\tau(a+),\tau(b-)]\subsetneq(0,1)$.

To construct ϕ, first define an inner function of $\mathbb{R}^{2+}$ (the upper half-plane), $B:\mathbb{R}^{2+}\circlearrowleft$, by

$$B(z):=\sum_{n=1}^{\infty}\frac{1}{2^{n+1}}\left(\frac{1+s_nz}{s_n-z}+\frac{1+t_nz}{t_n-z}\right)$$

with $s_n:=-\frac{1}{n}$ and $t_n=1+\frac{1}{n}$; then B is an inner function of $\mathbb{R}^{2+}$ and $B(i)=i$, $B:(0,1)\to\mathbb{R}$ is continuous, increasing with $B(0+),B(1-)\in\mathbb{R}$. Let $\psi:\mathbb{D}\to\mathbb{R}^{2+}$, $\psi(z)=i\frac{1+z}{1-z}$. Then $\phi:=\psi^{-1}\circ B\circ\psi$ is as advertised with $a=1/2$ and $b=3/4$.

2.7. Radial limit set. The *radial limit set* of ϕ is

$$\Lambda_\phi:=\{\beta\in\partial\mathbb{D}:\phi(r\beta)\xrightarrow[r\to1-]{}\phi(\beta)\in\partial\mathbb{D}\}.$$

For example, if $\phi(z):=\exp\left[-\frac{1+z}{1-z}\right]$, then

$$\phi(r\xi)\xrightarrow[r\to1-]{}\begin{cases}e^{-i\cot(\theta/2)}, & \xi=\chi(\theta)\neq1,\\ 0, & \xi=1,\end{cases}$$

and $\Lambda_\phi=\partial\mathbb{D}\setminus\{1\}$.

2.7.1. *Angular derivatives.* The inner function $\phi:\mathbb{D}\circlearrowleft$ has an *angular derivative* $\beta\in\mathbb{C}$ at $\xi\in\Lambda_\phi$ if there exists $\beta\in\mathbb{C}$ such that

$$\frac{\phi(z)-\phi(\xi)}{z-\xi}\xrightarrow[z\to\xi]{\angle}\beta,\quad\text{that is,}\quad\frac{\phi(z)-\phi(\xi)}{z-\xi}\xrightarrow[z\to\xi,\,|\xi-z|\le K(1-|z|)]{}\beta\quad\forall K>0.$$

Denote the angular derivative at ξ by $\beta=:\phi'_\angle(\xi)$.

Proposition 2.6 ([Sak07, §3]). *Let $\phi:\mathbb{D}\circlearrowleft$ be inner with $\tau=\tau(\phi)$. The following are equivalent for $\xi=\chi(x)\in\partial\mathbb{D}$:*

- *ϕ has an angular derivative at ξ;*
- *$\phi'(z)\xrightarrow[z\to\xi]{\angle}\beta\in\mathbb{C}$ and in this case $\beta=\phi'_\angle(\xi)$;*
- *$\int_{\mathbb{T}}\frac{d\nu_w(t)}{|\xi-\chi(t)|^2}<\infty$ for some (hence all) $w\in\mathbb{T}$, $w\neq x$.*

Proposition 2.7 ([Ale87], also [Sak07, Theorem 9.6]). *Let $\phi:\mathbb{D}\circlearrowleft$ be inner and let $E\in\mathcal{B}(\mathbb{T})$ with $m(E)>0$. Then $\tau=\tau(\phi)$ is locally invertible on E iff ϕ has an angular derivative at $\chi(x)$ for m-a.e. $x\in E$.*

Proof. $\Rightarrow$ Local invertibility on E entails forward nonsingularity on E, whence, by [Hei77], existence of angular derivatives a.s. on E.

$\Leftarrow$ Suppose that ϕ has an angular derivative at $\chi(x)$ for m-a.e. $x\in E$. By [Cra91, Lemma 1.5], τ is *almost uniformly differentiable* on E in the sense that there are $E_k\in\mathcal{B}(\mathbb{T})$ with $E_k\uparrow E$ mod m such that for all k and $\varepsilon>0$

there exists $\delta = \delta(k, \varepsilon) > 0$ such that

$$(15) \qquad |\tau(x) - \tau(y) - (x-y)g(x)| \le \varepsilon|x-y| \quad \forall x, y \in E_k,\ |x-y| < \delta,$$

where $g(x) = |\phi'(\chi(x))|$ with $\phi'(\chi(x))$ the angular derivative of ϕ at $\chi(x)$.

To see that τ is forward nonsingular on E, we note that by possibly shrinking the E_k (as in (15)), we may assume in addition that there exists $M_k > 0$ ($k \ge 1$) such that $g \le M_k$ on E_k.

Suppose that $k \ge 1$ and $S \in \mathcal{B}(E_k)$ with $m(S) = 0$, and fix $\varepsilon > 0$. There are intervals $\{I_n : n \ge 1\}$ such that

$$S \subset \bigcup_{n\ge1} I_n, \qquad |I_n| \le \delta(k, 1), \qquad \sum_{n\ge1} |I_n| < \frac{\varepsilon}{M_k + 1}.$$

By (15),

$$|\tau(x) - \tau(y)| \le (g(x)+1)|x-y| \le (M_k+1)|x-y|$$

and $m(\tau(E_k \cap I_n)) \le (M_k+1)m(I_n)$.

It follows that $\tau(S) \subset \bigcup_{n\ge1} \tau(E_k \cap I_n)$, whence

$$m(\tau(S)) \le \sum_{n\ge1} m(\tau(E_k \cap I_n) \le (M_k+1)\sum_{n\ge1} m(I_n) < \varepsilon$$

and thus $m(\tau(S)) = 0$.

For $S \in \mathcal{B}(E)$ with $m(S) = 0$,

$$m(\tau(S)) \xleftarrow[k\to\infty]{} m(\tau(E_k \cap S)) = 0$$

and τ is forward nonsingular on E, whence locally invertible on E by Theorem 2.2. ∎

An example of a probability preserving restriction which is "a.e. continuum-to-one" (a.e. Clark measure is nonatomic) was constructed in [Don65] (see also [Sak07, Ex. 9.7]).

In particular, for inner functions ϕ, $\tau(\phi)$ is locally invertible on $\mathbb{T}$ iff ϕ has an angular derivative at a.e. point on $\partial\mathbb{D}$.

The next result shows that this property is independent of the singularity set.

Proposition 2.8. *Let $E \subseteq \mathbb{T}$ be a closed set. Then there exists an inner function $\phi : \mathbb{D} \circlearrowleft$ with locally invertible restriction $\tau = \tau(\phi)$ such that $\phi(0) = 0$ and $\mathfrak{s}_\phi = \chi(E)$.*

Proof. Let $\Gamma \subset \mathbb{T}$ be countable with $\Gamma' = E$. We will construct ϕ via a Clark measure.

First fix $\varepsilon : \Gamma \to \mathbb{R}_+$ such that $\sum_{\gamma\in\Gamma} \varepsilon(\gamma) < \infty$ and then fix $\pi \in \mathcal{P}(\Gamma)$ with

$$\sum_{\gamma\in\Gamma} \frac{\pi_\gamma}{\varepsilon(\gamma)^2} < \infty.$$

We claim that

$$\int_{\mathbb{T}} \frac{d\pi(t)}{|t-x|^2} < \infty \quad \text{for } m\text{-a.e. } x \in \mathbb{T}, \tag{16}$$

where $\pi := \sum_{\gamma \in \Gamma} \pi_\gamma \delta_\gamma \in \mathcal{P}(\mathbb{T})$.

Indeed, since

$$\sum_{\gamma \in \Gamma} m(B(\gamma, \varepsilon(\gamma))) = 2 \sum_{\gamma \in \Gamma} \varepsilon(\gamma) < \infty$$

with $B(x,\varepsilon) := (x-\varepsilon, x+\varepsilon)$, by the Borel–Cantelli lemma there exists $K \in \mathcal{B}(\mathbb{T})$ with $K \cap \Gamma = \emptyset$ and $m(K) = 1$ such that for all $x \in K$, there exists $\Gamma_0(x) \in \Gamma$ finite such that

$$|x-\gamma| \geq \varepsilon(\gamma) \quad \forall \gamma \notin \Gamma_0(x). \tag{17}$$

Let $x \in K$. Then

$$\int_{\mathbb{T}} \frac{d\pi(t)}{|t-x|^2} = \sum_{\gamma \in \Gamma} \frac{\pi_\gamma}{|x-\gamma|^2} = \Big(\sum_{\gamma \in \Gamma_0(x)} + \sum_{\gamma \notin \Gamma_0(x)} \Big) \frac{\pi_\gamma}{|x-\gamma|^2}.$$

Since $K \subset \mathbb{T} \setminus \Gamma$, we have

$$\sum_{\gamma \in \Gamma_0(x)} \frac{\pi_\gamma}{|x-\gamma|^2} < \infty \quad \forall x \in K$$

and

$$\sum_{\gamma \notin \Gamma_0(x)} \frac{\pi_\gamma}{|x-\gamma|^2} \leq \sum_{\gamma \notin \Gamma_0(x)} \frac{\pi_\gamma}{\varepsilon(\gamma)^2} < \infty,$$

proving (16).

Next, we define $F : \mathbb{D} \to \mathbb{C}$ by

$$F(z) := \int_{\mathbb{T}} \frac{\chi(t)+z}{\chi(t)-z}\, d\pi(t).$$

Note that

$$\mathtt{Re}\left(\frac{\chi(t)+z}{\chi(t)-z}\right) = \frac{1-|z|^2}{|\chi(t)-z|^2} > 0 \quad \forall z \in \mathbb{D},$$

so $F : \mathbb{D} \to \mathbb{R}_+ \times \mathbb{R}$. Moreover, since $\pi(\mathbb{T}) = 1$, we have $F(0) = 1$.

To construct the inner function, define $\phi := \frac{F-1}{F+1} : \mathbb{D} \to \mathbb{C}$. Since $F(0) = 1$, we have $\phi(0) = 0$ and since $\mathtt{Re}\, F > 0$ on $\mathbb{D}$, it follows that $\phi : \mathbb{D} \circlearrowleft$.

Since $\pi \perp m$, we see that for m-a.e. $x \in \mathbb{T}$,

$$\lim_{r \to 1-} F(r\chi(x)) =: F(\chi(t)) \in i\mathbb{R},$$

whence for such $x \in \mathbb{T}$ we have

$$\phi(r\chi(x)) \xrightarrow[r \to 1-]{} \frac{F(\chi(x))-1}{F(\chi(x))+1} \in \partial\mathbb{D}$$

and $\phi : \mathbb{D} \circlearrowleft$ is inner.

Moreover, $\nu_0 = \pi$, whence by (16) and Proposition 2.6, ϕ has an angular derivative at $\chi(t)$ for a.e. $t \in \mathbb{T}$. By Proposition 2.7, $\tau(\phi)$ is locally invertible.

To finish, we note that $\mathfrak{s}_\phi = \mathtt{spt}\,\pi' = \Gamma' = \chi(E)$. ∎

The Baire category situation is different:

PROPOSITION 2.9. *Suppose that the inner function $\phi : \mathbb{D} \circlearrowleft$ has an angular derivative at a residual set of points. Then $\mathfrak{s}_\phi$ is nowhere dense.*

Proof. Fix $w \in \mathbb{T}$ and define $F : \mathbb{T} \to (0, \infty]$ by

$$F(x) := \int_{\mathbb{T}} \frac{d\nu_w(t)}{|\chi(x) - \chi(t)|^2}$$

and, for $r \in (0,1)$, define $F_r : \mathbb{T} \to \mathbb{R}_+$ by

$$F_r(x) := \int_{\mathbb{T}} \frac{d\nu_w(t)}{|r\chi(x) - \chi(t)|^2}.$$

Then for each $0 < r < 1$, $F_r : \mathbb{T} \to \mathbb{R}_+$ is continuous and $F_r(x) \xrightarrow[r \to 1-]{} F(x) \in (0, \infty]$ for all $x \in \mathbb{T}$, by dominated convergence in case $F(x) < \infty$ and by Fatou's lemma if $F(x) = \infty$.

By Baire's simple limit theorem, the set

$$C_\infty := \{x \in \mathbb{T} : F : \mathbb{T} \to (0, \infty] \text{ is continuous at } x\}$$

is residual in $\mathbb{T}$. By assumption, $\Lambda := [F < \infty]$ is residual, whence so is $C := C_\infty \cap \Lambda$. Thus for every $x \in C$, there exists $0 < a_x < b_x$ and an open interval $J_x \ni x$ such that

$$a_x < F < b_x \quad \text{on } J_x.$$

By Seidel's theorem [Sei34] (see also [Zyg02, Thm. 7.48]),

$$\mathfrak{s}_\phi \cap \chi(J_x) = \emptyset,$$

whence $\mathfrak{r}_\phi$ is open and dense and $\mathfrak{s}_\phi$ is nowhere dense. ∎

3. Quasicompactness of transfer operators. Let $\mathcal{L}$ be a Banach space. An operator $P \in \hom(\mathcal{L}, \mathcal{L})$ is called *quasicompact* if there exists $A = A(P) \in \hom(\mathcal{L}, \mathcal{L})$ of the form

$$A = \sum_{k=1}^{N} \lambda_k E_k$$

with $N \geq 1$, $E_1, \ldots, E_N \in \hom(\mathcal{L}, \mathcal{L})$ finite-dimensional projections, and $\lambda_1, \ldots, \lambda_N \in \mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$ such that the spectral radius satisfies

$$\rho(P - A) := \lim_{n \to \infty} \|(P - A)^n\|_{\hom(\mathcal{L}, \mathcal{L})}^{1/n} < 1.$$

Let (X, m, T) be a nonsingular transformation with transfer operator $\widehat{T} : L^1(m) \circlearrowleft$. We look for Banach spaces $\mathcal{L} \subset L^1(m)$ on which $\widehat{T} : \mathcal{L} \circlearrowleft$ acts quasicompactly.

If (X, m, T) is a weakly mixing, probability preserving transformation with transfer operator $\widehat{T}$ acting quasicompactly on $\mathcal{L} \subset L^1(m)$, then $A(\widehat{T})f = \mathbb{E}(f)$ and for all $\theta \in (\rho(\widehat{T} - \mathbb{E}), 1)$ there exists $M > 0$ such that

$$\|\widehat{T}^n f - \mathbb{E}(f)\|_{\mathcal{L}} \le M\theta^n \|f\|_{\mathcal{L}} \quad \forall f \in \mathcal{L}. \tag{18}$$

The property (18) is called *exponential decay of correlations* because it entails

$$\Big|\int_X u \cdot v \circ T^n \, dm - \mathbb{E}(u)\mathbb{E}(V)\Big| \le M\theta^n \|u\|_{\mathcal{L}} \|v\|_{L^1(m)}.$$

3.1. Doeblin–Fortet operators on an adapted pair. Let $\mathcal{L} \subset \mathcal{C} \subset L^1(m)$ be Banach spaces such that $(\mathcal{C}, \mathcal{L})$ is an *adapted pair* in the sense that

- $\|\cdot\|_{L^1(m)} \le \|\cdot\|_{\mathcal{C}} \le \|\cdot\|_{\mathcal{L}}$, $(\overline{\mathcal{L}})_{L^1(m)} = L^1(m)$, and $\mathcal{L}$-closed, bounded sets are $\mathcal{C}$-compact.

For example both $(L^1(m), \mathtt{Lip}(\mathbb{T}))$ and $(L^1(m), \mathtt{BV}(\mathbb{T}))$ are adapted pairs. Here $\mathtt{Lip}(\mathbb{T})$ denotes the Lipschitz functions on $\mathbb{T}$ (equivalently, the absolutely continuous functions with essentially bounded derivative), with norm

$$\|f\|_{\mathtt{Lip}} := \|f\|_1 + \|f'\|_\infty;$$

$\mathtt{Lip}$-closed, bounded sets are L^1-compact by the Arzelà–Ascoli theorem. Further, $\mathtt{BV}(\mathbb{T})$ denotes the functions of bounded variation on $\mathbb{T}$ with norm

$$\|f\|_{\mathtt{BV}} := \|f\|_1 + \bigvee f, \quad \text{where}$$

$$\bigvee f := \sup\Big\{\sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)| : 0 < t_1 < \dots < t_n = 1\Big\};$$

$\mathtt{BV}$-closed, bounded sets are L^1-precompact by Helly's theorem.

As in [Nor72, Ch. 3], we say that an operator $P \in \hom(\mathcal{L}, \mathcal{L}) \cap \hom(\mathcal{C}, \mathcal{C})$ is *Doeblin–Fortet* (D-F) on $(\mathcal{C}, \mathcal{L})$ if

$$\|P^n f\|_{\mathcal{C}} \le H\|f\|_{\mathcal{C}} \quad \forall n \in \mathbb{N},\ f \in L^1(m), \tag{19}$$

$$\exists \kappa \ge 1 : \ \|P^\kappa f\|_{\mathcal{L}} \le \theta\|f\|_{\mathcal{L}} + R\|f\|_{\mathcal{C}} \quad \forall f \in \mathcal{L}. \tag{20}$$

where $R, H \in \mathbb{R}_+$ and $\theta \in (0, 1)$.

EXAMPLE 3.1 (Adler arc maps). Let $(\mathbb{T}, m = \mathtt{Leb}, T)$ be an Adler map. It is a well known folklore result that the transfer operator $\widehat{T}$ is D-F on the adapted pair $(L^1(m), \mathtt{Lip}(\mathbb{T}))$.

It is also D-F on $(L^1(m), \mathtt{BV}(\mathbb{T}))$ because by [Zwe98, Cor. 1] an Adler map satisfies the assumptions of [Ryc83, Prop. 1], which proves the D-F inequality on $(L^1(m), \mathtt{BV})$.

The following lemma is a well-known consequence of the Yosida–Kakutani mean ergodic theorem [YK41, Theorem 1]. See also [ITM50], [Nor72, Chapter 3], [HH01, PP90], [LY73, Theorem 1].

Lemma 3.2 (rigidity of Doeblin–Fortet operators). *Suppose that P is a Doeblin–Fortet operator on the adapted pair $(\mathcal{C}, \mathcal{L})$. If $f \in \mathcal{C}$ and $Pf = f$, then $f \in \mathcal{L}$.*

It is shown in [ITM50] (see also [Nor72, Ch. 3], [HH01, PP90]) that a D-F operator $P \in \hom(\mathcal{L}, \mathcal{L})$ has spectral radius $\rho(P) \leq 1$ and that, if $\rho(P) = 1$, then P is quasicompact.

3.1.1. *Quasicompactness and the central limit theorem.* If $\psi \in \mathcal{L}$ and $\mathbb{E}(\psi) = 0$, then by Leonov's theorem [Leo61],

$$\exists \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\psi_n^2) =: \sigma_\psi^2 \geq 0, \quad \text{where} \quad \psi_n := \sum_{k=0}^{n-1} \psi \circ T^k, \tag{21}$$
$$\text{with equality iff } \psi = g - g \circ T \text{ for some } g \in \psi.$$

If in addition $\sigma_\psi > 0$, then (see [Gor04]) the *stationary process* (X, m, T, ψ) satisfies the central limit theorem:

$$m\left(\left[\frac{\psi_n}{\sigma_\psi \sqrt{n}} \leq t\right]\right) \xrightarrow[n\to\infty]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-s^2/2} ds. \tag{22}$$

3.2. Hardy spaces. The *harmonization* of $f \in L^p(m)$ $(1 \leq p \leq \infty)$ is $\widetilde{f} : \mathbb{D} \to \mathbb{C}$ defined by

$$\widetilde{f}(z) := \int_{\mathbb{T}} p_z f \, dm,$$

where p_z is as in (4). The function $\widetilde{f}$ is harmonic in $\mathbb{D}$ and satisfies

$$\sup_{r\in(0,1)} \|\widetilde{f}(r\chi)\|_p = \|f\|_p.$$

It is classical that the Hardy spaces consist of harmonizations:

$$h^p(\mathbb{D}) := \Big\{F : \mathbb{D} \to \mathbb{C} \text{ harmonic} : \sup_{r\in(0,1)} \|F(r\chi)\|_p < \infty\Big\} = \{\widetilde{f} : f \in L^p(m)\},$$
$$H^p(\mathbb{D}) := \{f \in h^p : \widetilde{f} \text{ analytic on } \mathbb{D}\} \cong \{f \in L^p(\mathbb{T}) : \widehat{f}(n) = 0 \ \forall n < 0\}.$$

Let

$$\Lambda_{\widetilde{f}} := \Big[\exists \lim_{r\to 1-} \widetilde{f}(r\chi) =: \widetilde{f}(\chi) \in \mathbb{C}\Big].$$

Then by Fatou's theorem, $m(\Lambda_{\widetilde{f}}) = 1$ and $\widetilde{f}(\chi) = f$ a.e.

3.2.1. *Action of the transfer operator.* Let $\phi : \mathbb{D} \circlearrowleft$ be inner with restriction $\tau = \tau(\phi)$. Then (see [Ale87], and also [Sak07, Thm. 3.1]) $\widehat{\tau} H^p(\mathbb{D}) \subset H^p(\mathbb{D})$, and if $\phi(0) = 0$, then for $d \geq 1, \widehat{\tau}(\chi^d)$ is a polynomial in χ of degree at most d.

LEMMA 3.3 (Basic Proposition). *Let $\phi : \mathbb{D} \circlearrowleft$ be inner with $\phi^{(k)}(0) = 0$ for all $0 \leq k < \kappa$. Then for $N \geq 1$ we have $\widehat{\tau}^N(\chi^d) = 0$ for all $1 \leq d < \kappa^N$, and for $d \geq \kappa^N$,*

$$\widehat{\tau}^N(\chi^d) = \sum_{\ell=1}^{\lfloor d/\kappa^N \rfloor} a^{(N)}(d,\ell)\chi^\ell \quad \textit{where} \quad a^{(N)}(k,\ell) = \overline{\widehat{(\phi^{[N]})^\ell}(k)}.$$

Proof. Let $d \geq 1$ and write $\widehat{\tau}^N(\chi^d) = \sum_{\ell \in \mathbb{Z}} a^{(N)}(d,\ell)\chi^\ell$. Then

$$a^{(N)}(d,\ell) = \langle \widehat{\tau}^N(\chi^d), \chi^\ell \rangle = \langle \chi^d, \chi^\ell \circ \tau^N \rangle = \langle \chi^d, \phi^N(\chi)^\ell \rangle = \overline{\widehat{(\phi^N)^\ell}(d)} \ (^4).$$

Thus

$$a^{(N)}(d,\ell) = 0 \quad \text{unless} \quad \ell \geq 1 \& d \geq \kappa^N \ell. \ \blacksquare \tag{23}$$

3.3. Weighted Hilbert spaces. A (Hilbert space) *weight* is a sequence $\underline{w} \in \mathbb{R}_+^{\mathbb{N}_0}$ satisfying $1 = w(0) < w(1) < \cdots < w(n) \uparrow \infty$. The associated *weighted Hilbert space* is

$$\hslash_{\underline{w}} := \Big\{ f \in L^2(m) : \|f\|_{\underline{w}}^2 := \sum_{n \in \mathbb{Z}} w(|n|)|\widehat{f}(n)|^2 < \infty \Big\},$$

equipped with the inner product

$$\langle u, v \rangle_{\underline{w}} := \sum_{n \in \mathbb{Z}} w(|n|)\widehat{u}(n)\overline{\widehat{v}(n)}.$$

Classical examples

(i) For $w(n) = n^2, \hslash_{\underline{w}}$ is isomorphic to the Sobolev space
$$W^{1,2}(\mathbb{T}) := \{ f \in C(\mathbb{T}) : f \text{ a.c. } \& \ f' \in L^2(m) \}.$$

(ii) For $b > 1$, let $w_{1,b}(n) := b^n$. Then
$$\hslash_b = \hslash_{\underline{w}_{1,b}} \cong h^2(B_{\mathbb{C}}(0,b)^o).$$

We call a weight $\underline{w}$ *summable* if $\sum_{n \geq 1} \frac{1}{w(n)} < \infty$. Both examples above are summable.

PROPOSITION 3.4. *If $\underline{w}$ is a summable weight, then $(L^2(\mathbb{T}), \hslash_{\underline{w}})$ is an adapted pair.*

(^{4}) Note that here $\phi^{[N]} := \underbrace{\phi \circ \cdots \circ \phi}_{N \text{ times}}$, whereas $(\phi^{[N]})^\ell := \underbrace{\phi^{[N]} \ldots \phi^{[N]}}_{\ell \text{ times}}$.

Proof. We show that $B(R) := \{f \in \hbar_{\underline{w}} : \|f\|_{\underline{w}} \le R\}$ is strongly compact in $L^2(m)$. To see this, let $f_j \in B(R)$ $(j \ge 1)$. Then for $n \in \mathbb{Z}$ and $j \ge 1$,

$$|\widehat{f_j}(n)| \le \frac{\|f_j\|_{\underline{w}}}{\sqrt{w(|n|)}} \le \frac{R}{\sqrt{w(|n|)}}$$

and there exist $j_\ell \to \infty$ and $a \in \ell^2(\mathbb{Z})$ such that

$$\widehat{f_{j_\ell}}(n) \xrightarrow[\ell\to\infty]{} a(n).$$

We claim that $\sum_{n\in\mathbb{Z}} w(|n|)|a(n)|^2 \le R^2$. Indeed,

$$R^2 \ge \sum_{n\in\mathbb{Z}} w(|n|)|\widehat{f_{j_\ell}}(n)|^2 \le \sum_{|n|\ge N} w(|n|)|\widehat{f_{j_\ell}}(n)|^2 \quad \forall N \ge 1,$$
$$\xrightarrow[\ell\to\infty]{} \sum_{|n|\le N} w(|n|)|a(n)|^2 \xrightarrow[N\to\infty]{} \sum_{n\in\mathbb{Z}} w(|n|)|a(n)|^2.$$

Let $A := \sum_{n\in\mathbb{Z}} a(n)\chi^n \in \hbar_{\underline{w}}$ with $\|A\|_{\underline{w}} \le R$.

To see that $f_\ell \xrightarrow[\ell\to\infty]{L^2(m)} A$, by the Riesz–Fischer theorem we get

$$\|f_{j_\ell} - A\|^2_{L^2(m)} = \sum_{n\in\mathbb{Z}} |\widehat{f_{j_\ell}}(n) - a(n)|^2 \xrightarrow[\ell\to\infty]{} 0$$
$$(\because\ 0 \xleftarrow[\ell\to\infty]{} |\widehat{f_{j_\ell}}(n) - a(n)|^2 \ll 1/w(|n|)). \ \blacksquare$$

The rest of this section is devoted to showing that the transfer operators of probability preserving, non-Möbius inner functions act quasicompactly on certain weighted Hilbert spaces.

Ivrii and Urbański [IU23] obtained spectral gaps for the action of $\widehat{\tau}$ on $W^{1,2}(\mathbb{T})$ ($\tau = \tau(\phi)$ with ϕ inner, $\phi(0) = 0$) and we obtain them on $\hbar_b$ (Proposition 3.5 below). In both cases, the minimal essential radius (as in §3.3.2) is the "Koenigs eigenvalue" $|\phi'(0)|$ (Proposition 3.7 below). However, we obtain superexponential decay of correlations on (e.g.) $\hbar_b$ when $\phi'(0) = 0$ (Proposition 3.6 below).

3.3.1. *Admissible weighted Hilbert spaces.* Call a summable weight $\underline{w} \in \mathbb{R}_+^{\mathbb{N}}$ and its associated weighted Hilbert space $\hbar_{\underline{w}}$ *admissible* if

$$\exists C = C_{\underline{w}} > 0 :\ W\left(\left\lfloor \frac{n}{K} \right\rfloor\right) \le C\frac{w(n)}{w(K)} \quad \forall n \ge K \ge 1, \tag{24}$$

where $W(n) := \sum_{k=1}^{n} w(k)$.

We call any $C_{\underline{w}}$ satisfying (24) an *admissibility constant* for $\underline{w}$.

For example, for $b > 1$, $w_{1,b}(n) = b^n$ defines an admissible weight with e.g. $C_{\underline{w}_{1,b}} = \frac{b}{b-1}$. Also, for $b, s > 1$, $\underline{w}_{s,b}$ defined by $w_{s,b}(n) := b^{n^s}$ defines an admissible weight.

On the other hand, for $t > 0$, $\underline{v}_t$ defined by $v_t(n) := n^t$ is not admissible (although summable for $t > 1$).

Recall from [Kat04, Def. I.2.10] that a Banach space $B \subset L^1(\mathbb{T}, m)$ is *homogeneous* if

$$\begin{aligned} &f \in B,\ s \in \mathbb{T} \implies f_s \in B,\ \|f_s\|_B = \|f\|, \\ &\|f - f_s\|_B \xrightarrow[s\to 0]{} 0 \quad \text{with } f_s(x) := f(x-s). \end{aligned} \tag{25}$$

Consequently (by [Kat04, Thm. I.2.11]), if B is homogeneous, then for $f \in B$,

$$f * p_r \in B, \quad \|f * p_r\|_B = \|f\|_B \quad \forall 0 < r < 1 \quad \& \quad f * p_r \xrightarrow[r\to 1]{B} f. \tag{26}$$

Any summably weighted Hilbert space $\hbar_{\underline{w}}$ is homogeneous.

Proposition 3.5 (Exponential decay of correlations). *Let $\phi : \mathbb{D} \circlearrowleft$ be inner, non-Möbius with $\phi(0) = 0$ and $\tau = \tau(\phi)$ and let $\underline{w} \in \mathbb{R}_+^{\mathbb{N}}$ be admissible.*

If there exists $0 < R = R_{\underline{w}} < 1$ such that $\sum_{n\ge 1} \frac{1}{R^{2n} w(n)} < \infty$, then for every $|\phi'(0)| < \rho < 1$ there exists $M > 0$ such that

$$\|\widehat{\tau}^N u - \mathbb{E}(u)\|_{\underline{w}} \le M\rho^N \|u\|_{\underline{w}} \quad \forall u \in \hbar_{\underline{w}},\ N \ge 0, \tag{27}$$

where $\mathbb{E}(u) := \int_{\mathbb{T}} u\, dm = \widehat{u}(0)$.

Proof. Fix $\rho \in (|\phi'(0)|, 1)$. We first show that

$$\forall 0 < r < 1,\ \exists M > 0 :\ |\phi^{[N]}(z)| \le M\rho^N\ \forall z \in \mathbb{D}, |z| \le r, \tag{28}$$

where $\phi^{[N]} = \underbrace{\phi \circ \cdots \circ \phi}_{N \text{ times}}$.

Proof of (28). By assumption, $\phi(z) = zg(z)$, where $g : \mathbb{D} \circlearrowleft$ is inner. Thus $\phi'(0) = g(0)$ and there exists $\mathfrak{r} = \mathfrak{r}_\rho$ such that

$$|g(z)| \le \rho \quad \& \quad |\phi(z)| \le \rho|z| \quad \forall |z| \le \mathfrak{r}.$$

Next, there exists N_ρ such that $|\phi^{[N_\rho]}(z)| \le \mathfrak{r}$ for $|z| \le r$, whence for $|z| \le r$,

$$|\phi^{[n]}(z)| \le \rho^{n-N_\rho}\mathfrak{r} \quad \forall n > N_\rho,$$

and (28) follows.

Proof of (27) *for $u \in \hbar_{\underline{w}} \cap H_0^2$.* For $u \in \hbar_{\underline{w}} \cap H_0^2$ set $v_n := \widehat{u}(n)/R^n$ where $R = R_{\underline{w}}$. Then

$$\sum_{n\ge 1} |v_n| = \sum_{n\ge 1} \sqrt{w(n)}|\widehat{u}(n)| \cdot \frac{1}{R^n\sqrt{w(n)}} \le \|u\|_{\underline{w}}^2 \sum_{n\ge 1} \frac{1}{R^{2n}w(n)} < \infty,$$

whence $v := \sum_{n\ge 1} v_n\chi^n \in C(\mathbb{T})$ and $v * p_R = u$.

Now

$$u = v * p_R = \int_{\mathbb{T}} v(t) p_{R\chi(t)}\, dm,$$

whence

$$u_N := \widehat{\tau}^N u = \int_{\mathbb{T}} v(t)\widehat{\tau}^N(p_{R\chi(t)})\,dm = \int_{\mathbb{T}} v(t)p_{\phi^{[N]}(R\chi(t))}\,dm \qquad \text{by (3)}$$

and

$$\begin{aligned}\widehat{u}_N(\ell) &= \int_{\mathbb{T}}\int_{\mathbb{T}} v(t)p_{\phi^{[N]}(R\chi(t))}\chi^{-\ell}\,dm\,dm\\ &= 1_{\mathbb{N}}(\ell)\int_{\mathbb{T}} v(t)\widehat{p}_{\phi^{[N]}(R\chi(t))}(\ell)\,dm\\ &= 1_{\mathbb{N}}(\ell)\int_{\mathbb{T}} v(t)(\overline{\phi^{[N]}(R\chi(t))})^{\ell}\,dm.\end{aligned}$$

Thus

$$|\widehat{u}_N(\ell)|^2 \le 1_{\mathbb{N}}(\ell)\int_{\mathbb{T}}|v|^2\,dm\int_{\mathbb{T}}|\phi^{[N]}(R\chi(t))|^{2\ell}\,dt \le \|u\|^2_{\underline{w}}(M\rho^N)^{2\ell}.$$

where $M > 0$ is as in (28).

Let N_0 be such that $\mathfrak{b}_n := M^2\rho^{2n} < \frac{1}{2}$ for $n \ge N_0$. For $N \ge N_0$,

$$\|\widehat{\tau}^N u\|^2_b = \sum_{\ell\ge1} b^{\ell}|\widehat{u_N}(\ell)|^2 \le \|u\|^2_b\sum_{\ell\ge1}\mathfrak{b}^{\ell}_N = \frac{\mathfrak{b}_N}{1-\mathfrak{b}_N}\|u\|^2_b \le 2M^2b\rho^{2N}\|u\|^2_b,$$

proving (27).

To continue, let $w \in \hbar_{\underline{w}}$. Then

$$w = F + \overline{G} + \mathbb{E}(w) \qquad \text{for some } F, G \in \hbar_{\underline{w}} \cap H^2_0$$

and

$$\widehat{\tau}^N w = \widehat{\tau}^N F + \overline{\widehat{\tau}^N G} + \mathbb{E}(w).$$

By (27) for F and G,

$$\begin{aligned}\|\widehat{\tau}^N w - \mathbb{E}(w)\|^2_b &= \|\widehat{\tau}^N F\|^2_b + \|\widehat{\tau}^N G\|^2_b \le M\rho^{2N}(\|F\|^2_b + \|G\|^2_b)\\ &\le M\rho^{2N}\|w\|^2_{\underline{w}}. \ \blacksquare\end{aligned}$$

Proposition 3.6 (Superexponential decay of correlations). *Suppose that $\phi(z) = z^{\kappa}\Phi(z)$ with $\kappa > 1$ and $\Phi : \mathbb{D} \circlearrowleft$ inner and let $\underline{w} \in \mathbb{R}^{\mathbb{N}}_+$ be admissible. Then*

$$\|\widehat{\tau}^N u - \mathbb{E}(u)\|_{\underline{w}} \le \frac{\sqrt{C_{\underline{w}}}}{\sqrt{w(\kappa^N)}}\|u\|_{\underline{w}} \qquad \forall u \in \hbar_{\underline{w}},\ N \ge 0. \tag{29}$$

Proof. By Lemma 3.3, for $N, k \ge 1$,

$$\widehat{\tau}^N(\chi^k) = \sum_{\ell=1}^{\lfloor d/\kappa^N\rfloor} a^{(N)}(k,\ell)\chi^{\ell},$$

where $a^{(N)}(k,\ell) = \overline{\widehat{(\phi^N)^\ell}(k)}$ and $\sum_{\ell=1}^{0} = 0$. Thus also $\sum_{k\ge\kappa^N\ell} |a^{(N)}(k,\ell)|^2 = 1$.

Proof of (29) *for* $u \in \hbar_{\underline{w}} \cap H_0^2$. We have

$$\begin{aligned}\widehat{\tau}^N u &= \sum_{k\ge1} u_k \widehat{\tau}^N(\chi^k) \quad (\text{where } u_k = \widehat{u}(k))\\ &= \sum_{\ell\ge1,\, k\ge\kappa^N\ell} u_k a^{(N)}(k,\ell)\chi^\ell\\ &= \sum_{\ell\ge1}\Big(\sum_{k\ge\kappa^N\ell} u_k a^{(N)}(k,\ell)\Big)\chi^\ell.\end{aligned}$$

Thus, using Cauchy–Schwarz and (23), we get

$$|(\widehat{\tau}^N u)_\ell|^2 = \Big|\sum_{k\ge\kappa^N\ell} u_k a^{(N)}(k,\ell)\Big|^2 \le \sum_{k\ge\kappa^N\ell} |u_k|^2$$

and

$$\begin{aligned}\|\widehat{\tau}^N u\|_{\underline{w}}^2 &= \sum_{\ell\ge1} w(\ell)|(\widehat{\tau}^N u)_\ell|^2 \le \sum_{\ell\ge1} w(\ell) \sum_{k\ge\kappa^N\ell} |u_k|^2\\ &\le \sum_{k\ge\kappa^N} |u_k|^2 \sum_{1\le\ell\le k/\kappa^N} w(\ell) = \sum_{k\ge\kappa^N} |u_k|^2 W(\lfloor k/\kappa^N\rfloor)\\ &= \sum_{k\ge\kappa^N} |u_k|^2 w(k)\cdot\frac{W(\lfloor k/\kappa^N\rfloor)}{w(k)} \le \frac{C_{\underline{w}}}{w(\kappa^N)}\|u\|_{\underline{w}}^2\end{aligned}$$

by (24), as desired.

To continue, let $u \in \hbar_{\underline{w}}$. Then

$$u = F + \overline{G} + \mathbb{E}(u) \quad \text{for some } F, G \in \hbar_{\underline{w}} \cap H_0^2$$

and

$$\widehat{\tau}^N u = \widehat{\tau}^N F + \overline{\widehat{\tau}^N G} + \mathbb{E}(u).$$

By the previous step,

$$\begin{aligned}\|\widehat{\tau}^N u - \mathbb{E}(u)\|_{\underline{w}}^2 &= \|T^N F\|_{\underline{w}}^2 + \|\widehat{\tau}^N G\|_{\underline{w}}^2\\ &\le \frac{C_{\underline{w}}}{w(\kappa^N)}(\|F\|_{\underline{w}}^2 + \|G\|_{\underline{w}}^2) = \frac{C_{\underline{w}}}{w(\kappa^N)}\|u\|_{\underline{w}}^2. \quad ■\end{aligned}$$

3.3.2. *Essential spectral radius.* For $(\mathbb{T}, m, T)$ a weakly mixing, probability preserving transformation, suppose that $\widehat{T}$ is a Doeblin–Fortet operator on the adapted pair $(L^1(\mathbb{T}, m), \mathcal{L})$.

The *essential spectral radius* of $\widehat{T} : \mathcal{L} \circlearrowleft$ is

$$\rho_{\mathsf{ess}}(\widehat{T}, \mathcal{L}) := \rho(T, \mathcal{L}_0) = \lim_{n\to\infty} \|\widehat{T}^n\|_{\hom(\mathcal{L}_0,\mathcal{L}_0)}^{1/n},$$

where $\mathcal{L}_0 := \{f \in \mathcal{L} : \mathbb{E}(f) = 0\}$. Equivalently, $\rho(\widehat{T}, \mathcal{L})$ is the greatest lower bound of the collection of $\theta \in (0,1)$ satisfying (18).

PROPOSITION 3.7 (Minimal essential spectral radius ([5])). *Let $\phi : \mathbb{D} \circlearrowleft$ be non-Möbius, inner with $\phi(0) = 0$, and suppose that $\widehat{\tau}$ is a Doeblin–Fortet operator on the adapted pair $(L^1(m), \mathcal{L})$, where there exists $R > 0$ such that $p_z \in \mathcal{L}$ for all $z \in \mathbb{D}$ with $|z| < R$. Then $\rho_{\mathsf{ess}}(\widehat{\tau}, \mathcal{L}) \geq |\phi'(0)|$.*

Proof. By Koenigs' theorem ([Koe84], see also [Sha93, §6.1]), there exists $z \in \mathbb{D}$ with $|z| < R$ such that $|\phi^n(z)|^{1/n} \xrightarrow[n\to\infty]{} |\phi'(0)|$, and by (3), if $\widehat{\tau}$ satisfies (18) with constant θ on $\mathcal{L} \in \mathfrak{h}$, then

$$M\theta^n \geq \|\widehat{\tau}^n(p_z) - 1\|_{\mathcal{L}} \geq \|\widehat{\tau}^n(p_z) - 1\|_1 \geq \varepsilon|\phi^n(z)| = |\phi'(0)|^{n+o(n)}. \blacksquare$$

4. Perturbations and central limits. Let (X, m, T) be a weakly mixing, probability preserving transformation and let $\psi : X \to \mathbb{C}$ be measurable.

For $z \in \mathbb{C}$ such that $e^{z\psi} \in L^\infty(m)$, define the *perturbed* (or twisted) *transfer operator* $\mathit{\Pi}_{z,\psi} : L^1(m) \circlearrowleft$ by

$$\mathit{\Pi}_{z,\psi} f := \widehat{T}(e^{z\psi} f).$$

The operators $P_t = P_{t,\psi} := \widehat{T}(e^{it\psi} f)$ are also known as *characteristic function operators*, because $\mathbb{E}(P_t \mathbb{1}) = \mathbb{E}(e^{it\psi})$.

Let $\mathcal{L}$ be a Banach space of functions on $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is an *$\mathcal{L}$-multiplier* if $f \cdot u \in \mathcal{L}$ for all $u \in \mathcal{L}$. Let $M(\mathcal{L}) := \{\mathcal{L}\text{-multipliers}\}$. By the Resonance Theorem,

$$\|f\|_{M(\mathcal{L})} := \sup\{\|fu\|_{\mathcal{L}} : u \in \mathcal{L},\ \|u\|_{\mathcal{L}} = 1\} < \infty \quad \forall f \in M(\mathcal{L}).$$

Evidently, if $\mathbb{1} \in \mathcal{L}$, then $M(\mathcal{L}) \subseteq \mathcal{L}$. Indeed, for $\mathcal{L} = \mathtt{Lip}$ or $\mathtt{BV}$, $M(\mathcal{L}) = \mathcal{L}$.

In general, if $f \in M(\mathcal{L})$, then $f^N \in M(\mathcal{L})$ for all $N \geq 1$, with $\|f^N\|_{M(\mathcal{L})} \leq \|f\|^N_{M(\mathcal{L})}$, and $e^{zf} \in M(\mathcal{L})$ for all $z \in \mathbb{C}$, with $\|e^{zf}\|_{M(\mathcal{L})} \leq e^{|z|\,\|f\|_{M(\mathcal{L})}}$.

THEOREM 4.1 (Nagaev's Theorem [Nag57, RE83] ([6])). *Suppose that $\widehat{T}$ is a Doeblin–Fortet operator on the adapted pair $(L^1(m), \mathcal{L})$ and that $\psi \in M(\mathcal{L}) \cap \mathcal{L}$ satisfies $\mathbb{E}(\psi) = 0$ and*

$$\begin{aligned} &\exists\varepsilon > 0 :\ P_{t,\psi} \in \hom(\mathcal{L}, \mathcal{L})\ \forall |t| < \varepsilon; \\ &t \mapsto P_{t,\psi} \text{ is } C^2 \text{ from } (-\varepsilon, \varepsilon) \text{ to } \hom(\mathcal{L}, \mathcal{L}). \end{aligned} \tag{30}$$

Then:

(i) *There exists $0 < \mathcal{E} < \varepsilon$ such that P_t is a Doeblin–Fortet operator on $(L^2(m), \mathcal{L})$ for $|t| < \mathcal{E}$.*

([5]) Cf. [BCJ23].

([6]) See also [PP90, HH01], [AD01, Lemma 4.2].

(ii) *There are constants* $K > 0$ *and* $\theta \in (0,1)$, *and* C^2 *functions* $\lambda : B(0,\mathcal{E}) \to B_{\mathbb{C}}(0,1)$ *and* $N : B(0,\mathcal{E}) \to \hom(\mathcal{L},\mathcal{L})$, *such that*

$$\|P_t^n h - \lambda(t)^n N(t)h\|_{\mathcal{L}} \le K\lambda(t)^n \|h\|_{\mathcal{L}} \quad \forall |t| < \mathcal{E},\ n \ge 1,\ h \in \mathcal{L}, \tag{31}$$

where for $|t| < \mathcal{E}$, $N(t)$ *is a projection onto a one-dimensional subspace.*

(iii) *If* $\sigma_\psi := \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\psi_n^2) > 0$ *as in* (21), *then* $\lambda(t) = 1 - \sigma_\psi^2 t^2 + o(t^2)$ *as* $t \to 0$ *and the* conditional central limit theorem *holds:*

$$\widehat{T}^n 1_{[\frac{\psi_n}{\sigma\sqrt{n}} \le t]} \xrightarrow[n\to\infty]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-s^2/2}\,ds. \tag{32}$$

Remarks. 1. If, in addition, P_t is a Doeblin–Fortet operator for all $t \in \mathbb{R}$ and ψ is T-*aperiodic* in the sense that $e^{it\psi} = \lambda g \circ \tau / g$ with $t \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$ & $g : \mathbb{T} \to \mathbb{C}$ entails $t = 0$, $\lambda = 1$ & g constant, then the conditional local limit theorem holds: for $I \subset \mathbb{R}$ an interval, and $k_n \in \mathbb{Z}$ with $\frac{k_n}{\sigma\sqrt{n}} \to \kappa \in \mathbb{R}$,

$$\sigma\sqrt{n}\widehat{T}^n(1_{[\Psi_n \in n\mathbb{E}(\Psi)+k_n+I]}) \xrightarrow[n\to\infty]{} \frac{|I|}{\sqrt{2\pi}} e^{-\kappa^2/2} \tag{33}$$

and the skew product $(X \times \mathbb{R}, m \times \mathtt{Leb}, T_\psi)$ is ergodic, where $T_\psi(x,y) = Tx, y + \psi(x))$. See [Gui89, GH88, AD01].

2. If (X, m, T) is an AFU map as in [Zwe98] (e.g. an Adler arc map), then for $\psi \in \mathtt{BV}$, the characteristic function operator $P_{t,\psi}$ is a Doeblin–Fortet operator on $(L^1(m), \mathtt{BV})$ for all $t \in \mathbb{R}$. See [ADSZ04, §5].

To obtain (32) for the stationary process (X, m, T, ψ) via Nagaev's theorem, we verify (30) and then $\sigma_\psi > 0$.

Theorem 4.2 (Analyticity of perturbation). *Suppose that* $\phi : \mathbb{D} \circlearrowleft$ *is non-Möbius inner with* $\phi(0) = 0$.

Let $b > 1$ *and let* $\underline{w}$ *be a summable weight* $\sum_{n\ge1} b^n/w(n) < \infty$.

If $\psi \in \hbar_{\underline{w}}$ *then for all* $z \in \mathbb{C}$,

$$\Pi_z := \Pi_{z,\psi} \in \hom(\hbar_b, \hbar_b).$$

Moreover, $\mathbb{C} \ni z \mapsto \Pi_z \in \hom(\hbar_b, \hbar_b)$ *is holomorphic with*

$$\frac{d^n \Pi_z}{d\,z^n}(f) = \Pi_z(\psi^n f) =: \Pi_z^{(n)}(f).$$

In particular, (30) *holds.*

Lemma 4.3 (Multiplier lemma).

(i) *Let* $b > 1$ *and let* $\underline{w}$ *be a summable weight with* $\sum_{n\ge1} b^n/w(n) < \infty$. *Then* $\hbar_{\underline{w}} \subseteq M(\hbar_b)$ *with*

$$\|f\|_{M(\hbar_b)} \le R_{b,\underline{w}} \|f\|_{\underline{w}} \quad \forall f \in \hbar_{\underline{w}}, \tag{34}$$

where $R_{b,\underline{w}} := \sqrt{1 + 2\sum_{n\ge1} b^n/w(n)}$.

(ii) *If $\underline{w}$ is a weight with $\overline{\lim}_{n\to\infty} w(n+1)/w(n) = \infty$, then there are no non-constant multipliers of $\hbar_{\underline{w}}$.*

Note that if $w(n) = B^n$ with $B > b$ then $R_{b,\underline{w}} = \sqrt{\frac{B+b}{B-b}}$.

Proof of (i). We have

$$\begin{aligned}
\|fg\|_{\hbar_b}^2 &= \sum_{n\in\mathbb{Z}} b^{|n|}|\widehat{fg}(n)|^2 = \sum_{n\in\mathbb{Z}} b^{|n|}\Big|\sum_{k\in\mathbb{Z}} \widehat{f}(k)\widehat{g}(n-k)\Big|^2 \\
&= \sum_{n\in\mathbb{Z}} b^{|n|}\Big|\sum_{k\in\mathbb{Z}} \sqrt{w(|k|)}\big|\widehat{f}(k)\big|\frac{|\widehat{g}(n-k)|}{\sqrt{w(|k|)}}\Big|^2 \\
&\le \sum_{n\in\mathbb{Z}} b^{|n|}\sum_{k\in\mathbb{Z}} w(|k|)|\widehat{f}(k)|^2 \sum_{\ell\in\mathbb{Z}} \frac{|\widehat{g}(n-\ell)|^2}{w(|\ell|)} \\
&= \|f\|_{\underline{w}}^2 \sum_{n,\ell\in\mathbb{Z}} b^{|n|}\frac{|\widehat{g}(n-\ell)|^2}{w(|\ell|)} = \|f\|_{\underline{w}}^2 \sum_{n,\ell\in\mathbb{Z}} \frac{b^{|\ell|}}{w(|\ell|)} b^{|n|-|\ell|}|\widehat{g}(n-\ell)|^2 \\
&\le \|f\|_{\underline{w}}^2 \sum_{n,\ell\in\mathbb{Z}} \frac{b^{|\ell|}}{w(|\ell|)} b^{|n-\ell|}|\widehat{g}(n-\ell)|^2 = \|f\|_{\underline{w}}^2\|g\|_{\hbar_b}^2 \sum_{\ell\in\mathbb{Z}} \frac{b^{|\ell|}}{w(|\ell|)} \\
&= R_{b,\underline{w}}^2\|f\|_{\underline{w}}^2\|g\|_{\hbar_b}^2.
\end{aligned}$$

Proof of (ii). Suppose otherwise. Then there exist $f \in M(\hbar_{\underline{w}})$ and $\ell \ge 1$ with $\widehat{f}(\ell) \ne 0$. Suppose that $\nu_k \uparrow \infty$ with $w(\nu_k+1) \ge kw(\nu_k)$ for $k \ge 1$. It follows that

$$\infty > \|f\|_{M(\hbar_{\underline{w}})}^2 \ge \frac{\|f\chi^{\nu_k}\|_{\hbar_{\underline{w}}}^2}{w(\nu_k)} \ge |\widehat{f}(\ell)|^2\frac{w(\ell+\nu_k)}{w(\nu_k)} \ge k|\widehat{f}(\ell)|^2 \xrightarrow[k\to\infty]{} \infty. \ \blacksquare$$

Proof of Theorem 4.2. It suffices to show that for every $\omega \in \mathbb{C}$, there exists $\varepsilon = \varepsilon_\omega > 0$ such that

$$\sum_{n=0}^{N} \frac{(z-\omega)^n}{n!}\varPi_\omega^{(n)} \xrightarrow[N\to\infty]{\mathrm{hom}(\hbar_b,\hbar_b)} \varPi_z \quad \forall z \in B(\omega,\varepsilon). \tag{35}$$

To this end, fix $\omega \in \mathbb{C}$ and $1 < b < b_1 < b_2$ with $\sum_{n\ge 1} b_2^n/w(n) < \infty$. Let $f \in \hbar_b$. Then for $k \ge 1$,

$$\begin{aligned}
\|\varPi_\omega^{(k)}(f)\|_{\hbar_b} &= \|\widehat{\tau}(\psi^k e^{\omega\psi} f)\|_{\hbar_b} \le M\rho\|\psi^k e^{\omega\psi} f\|_{\hbar_b} && \text{by (27)} \\
&\le M\rho R_{b,\underline{w}_{1,b_1}}\|f\|_{\hbar_b}\|\psi^k e^{\omega\psi}\|_{b_1} && \text{by (34)} \\
&= M\rho\sqrt{\frac{b_1+b}{b_1-b}}\|f\|_{\hbar_b}\|\psi^k e^{\omega\psi}\|_{b_1}.
\end{aligned}$$

To continue, by repeated application of (34),

$$\begin{aligned}\|\psi^k e^{\omega\psi}\|_{b_1} &\le \left(\sqrt{\frac{b_2+b_1}{b_2-b_1}}\right)^{k+1} \|\psi\|_{b_2}^k \|e^{\omega\psi}\|_{b_2} \\ &\le \left(\sqrt{\frac{b_2+b_1}{b_2-b_1}}\right)^{k+1} \|\psi\|_{b_2}^k \exp[|\omega| R_{b_2,\underline{w}} \|\psi\|_{\underline{w}}^2].\end{aligned}$$

Thus

$$\|\Pi_\omega^{(k)}\|_{\mathrm{hom}(\mathscr{k}_b,\mathscr{k}_b)} \ll \left(\sqrt{\frac{b_2+b_1}{b_2-b_1}}\right)^k$$

and (35) holds with $\varepsilon_\omega = \sqrt{\frac{b_2-b_1}{b_2+b_1}}$. ∎

4.1. Periodicity. Let $\phi : \mathbb{D} \circlearrowleft$ be non-Möbius inner with $\phi(0) = 0$ and let $\psi : \mathbb{T} \to \mathbb{R}$ be measurable.

We call $t \in \mathbb{R}$ a *τ-period* of ψ if there are $t \in \mathbb{R}$, $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $g : \mathbb{T} \to \mathbb{C}$ measurable such that $e^{it\psi} = \lambda g \circ \tau / g$.

We denote the collection of τ-periods of ψ by $\mathcal{Q}(\psi)$, and we call ψ *τ-aperiodic* if $\mathcal{Q}(\psi) = \{0\}$ and *τ-periodic* otherwise.

It is standard to show that for $t \in \mathbb{R}$, $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $f \in L^1(m)$,

$$e^{it\psi} f = \lambda f \circ \tau \iff P_t(f) := \widehat{\tau}(e^{it\psi} f) = \lambda f, \tag{36}$$

and also, if the limit $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\psi_n^2) =: \sigma_\psi^2 \ge 0$ exists, then $\sigma_\psi > 0$ if $\mathcal{Q}(\psi)$ is discrete.

THEOREM 4.4. *Let $\phi : \mathbb{D} \circlearrowleft$ be non-Möbius inner with with Denjoy–Wolff point in $\mathbb{D}$ and nonempty singularity set.*

- (i) *If $\psi : \mathbb{T} \to \mathbb{R}$ is nonconstant and real analytic, then $\mathcal{Q}(\psi)$ is discrete and the stationary process $(\mathbb{T}, \pi_{\mathfrak{d}_\phi}, \tau, \psi)$ satisfies* (32).
- (ii) *If both $\mathcal{Z}(\phi)$ and $\mathfrak{s}_\phi$ are finite, and $\psi : \mathbb{T} \to \mathbb{R}$ is nonconstant and of bounded variation, then ψ is τ-aperiodic and the stationary process $(\mathbb{T}, \pi_{\mathfrak{d}_\phi}, \tau, \psi)$ satisfies* (33).

The assumption $\mathfrak{s}_\phi \neq \emptyset$ is essential. If ϕ is a finite Blaschke product with $\mathfrak{d}(\phi) \in \mathbb{D}$ and $g : \mathbb{T} \to \mathbb{R}$ nonconstant and real analytic, then so is $\psi := g - g \circ \tau : \mathbb{T} \to \mathbb{R}$, whence $P_t(\chi(tg)) = \chi(tg)$ for all $t \in \mathbb{R}$ and $\mathcal{Q}(\psi) = \mathbb{R}$.

Proof of (i). Let $\mathcal{E} > 0$ be as in (i). Fix $1 < b < B$ with $\psi \in \mathscr{k}_B$. If $\mathcal{Q}(\psi)$ is not discrete, then there are $t \in (0, \mathcal{E}) \cap \mathcal{Q}(\psi)$, $f \in L^1(m)$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $P_t(f) = \lambda f$. For $t \in (0, \mathcal{E}) \cap \mathcal{Q}(\psi)$, P_t is a Doeblin–Fortet operator on $(L^2(m), \mathscr{k}_b)$. Write $Q := \overline{\lambda} P_t$. Then $Q : \mathscr{k}_b \circlearrowleft$ is a Doeblin–Fortet operator, $Qf = f$ and by Lemma 3.1, $f \in \mathscr{k}_b$.

By (36) we have

$$f(\tau x) = \overline{\lambda} f(x) e^{it\psi(x)} \quad \text{for } m\text{-a.e. } x \in \mathbb{T}. \tag{37}$$

Using harmonization, we see that (37) holds for all $x \in \Lambda_\phi$.

Now suppose that v is a point of discontinuity of τ. Then $\chi(v) \in \mathfrak{s}_\phi$.

By Seidel's theorem [Sei34] (see also [Zyg02, Thm. 7.48]), for every $w \in \mathbb{T}$ there exists $z_n = z_n(w) \in \Lambda_\phi$ such that $z_n \to v$ and $\tau(z_n) = w$. Thus

$$f(w) = f(\tau(z_n)) = \overline{\lambda} f(z_n)\psi(z_n) \xrightarrow[n\to\infty]{} \overline{\lambda} f(v)\psi(v),$$

so f is constant, $e^{it\psi} \equiv \lambda$, whence ψ (being continuous) is constant. ∎

Proof of (ii). Under the assumptions of (ii), $(\mathbb{T}, \pi_{\mathfrak{d}(\phi)}, \tau)$ is an Adler arc map, whence an AFU map as in [Zwe98] and $P_{t,\psi}$ is a Doeblin–Fortet operator on $(L^1(m), \mathtt{BV}(\mathbb{T}))$ for all $t \in \mathbb{R}$. Using [ADSZ04, §5], it suffices to prove that ψ is τ-aperiodic.

To this end, suppose that $f : \mathbb{T} \to \mathbb{C}$ is measurable and satisfies (37). By Lemma 3.1, $f \in \mathtt{BV}$. We must show that f is constant. Suppose otherwise; then there exist $\xi, \zeta \in \mathbb{T}$ such that $|f(\xi) - f(\zeta)| =: \eta > 0$.

Let $v \in \chi^{-1}\mathfrak{s}_\phi$. Then there exists $z_n \to v$ monotonically with $\tau(z_{2n}) = \xi$ and $\tau(z_{2n+1}) = \zeta$. The function $G(x) := \overline{\lambda} f(x) e^{it\psi(x)}$ is in $\mathtt{BV}$, thus so is $f \circ \tau = G$. Therefore

$$\infty > \bigvee f \circ \tau \geq \sum_{n\geq 1} |f(\tau(z_n)) - f(\tau(z_{n+1}))| = \sum_{n\geq 1} \eta = \infty. \ \blacksquare$$

5. A local version of Aleksandrov's theorem. Fix $A \in \mathcal{B}(\mathbb{T})$ with $m(A) > 0$ and a sub-σ-algebra $\mathcal{C} \subseteq \mathcal{B}(A)$.

For $1 \leq p \leq \infty$ write

$$H_0^p(A, \mathcal{C}) := \{f \in H_0^p : f|_A \text{ is } \mathcal{C}\text{-measurable}\}.$$

By [Rud74, Thm. 17.18] if $f, g \in H_0^2(m)$ and $f|_A \equiv g|_A$, then $f \equiv g$.

Let $P : H_0^2 \to H_0^2(A, \mathcal{C})$ be the orthogonal projection in the sense that

$$(\mathtt{Id} - P)H_0^2 \subset H_0^2(A, \mathcal{C})^\perp. \tag{38}$$

Call $(A, \mathcal{C})$ an *analytic pair* if

$$\mathbb{E}_{m_A}^{\mathcal{C}}(f) = (Pf)|_A \quad \text{for } f \in H_0^2,$$

where $m_A(B) := m(A \cap B)$ and $\mathbb{E}_{m_A}^{\mathcal{C}}(f)$ is conditional expectation on the measure space (A, m_A) with respect to the sub-σ-algebra $\mathcal{C} \subseteq \mathcal{B}(A)$.

Let $A \in \mathcal{B}(\mathbb{T})$ with $m(A) > 0$. It is easy to see that $(A, \mathcal{C})$ is an analytic pair if either $\mathcal{C} \overset{m_A}{=} \mathcal{B}(A)$ (in which case $H_0^2(A, \mathcal{C}) = H_0^2$), or $\mathcal{C} \overset{m_A}{=} \{\emptyset, A\}$ (in which case $H_0^2(A, \mathcal{C}) = \{0\}$).

Next, we give an example which turns out to be general.

EXAMPLE 5.1. If $A \in \tau^{-1}\mathcal{B}(\mathbb{T})$ and $\mathcal{C} = \tau^{-1}\mathcal{B}(\mathbb{T}) \cap A$, where $\tau = \tau(\phi)$ with $\phi : \mathcal{D} \circlearrowleft$ inner with $\phi(0) = 0$, then $(A, \mathcal{C})$ is an analytic pair.

Proof. We have

$$\begin{aligned} H_0^2(A, \tau^{-1}\mathcal{B}) &:= \{f \in H_0^2 : f|_A \text{ is } \tau^{-1}\mathcal{B}\text{-measurable}\} \\ &= \{f \in H_0^2 : \exists g \in H_0^2,\ f|_A = g \circ \tau|_A\} \\ &= H_0^2 \circ \tau \quad \text{by [Rud74, Thm. 17.18].} \end{aligned}$$

To continue, note that because $m \circ \tau^{-1} = m$,

$$\mathbb{E}_m^{\tau^{-1}\mathcal{B}(\mathbb{T})}(f) = \widehat{\tau}(f) \circ \tau.$$

Since $\widehat{\tau} : H_0^2 \circlearrowleft$,

$$P := \mathbb{E}_m^{\tau^{-1}\mathcal{B}(\mathbb{T})} = \widehat{\tau}(f) \circ \tau : H_0^2 \to H_0^2 \circ \tau$$

is an orthogonal projection.

Let $A = \tau^{-1}B$. Then

$$\mathbb{E}_{m_A}^{\mathcal{C}}(f) = \mathbb{E}_m^{\tau^{-1}\mathcal{B}}(1_B \circ \tau f)|_A = ((1_B \widehat{\tau} f) \circ \tau)|_A = (Pf)|_A. \ \blacksquare$$

THEOREM 5.2. *If $(A, \mathcal{C})$ is an analytic pair, then either $\mathcal{C} = \{\emptyset, A\}$, or there exists $\phi : \mathcal{D} \circlearrowleft$ inner with $\phi(0) = 0$ such that $A \in \tau^{-1}\mathcal{B}$ and $\mathcal{C} = A \cap \tau^{-1}\mathcal{B}$, where $\tau = \tau(\phi)$.*

The cases with $C = \mathbb{T}$ are established in [Ale86] ([7]).

Proof. We divide the proof into several claims.

CLAIM 1. *$P(gf) = gP(f)$ for all $f \in H_0^\infty$ and $g \in H_0^2(A, \mathcal{C})$.*

Proof. We have

$$P(gf)|_A = \mathbb{E}_{m_A}^{\mathcal{C}}(fg) = g|_A \mathbb{E}_{m_A}^{\mathcal{C}}(f) = (gP(f))|_A,$$

and by [Rud74, Thm. 17.18], $P(gf) = gP(f)$ a.s. on $\mathbb{T}$.

CLAIM 2. *For every $f \in L^2(A, \mathcal{C}, m_A)$ there are $g, h \in H_0^2(A, \mathcal{C})$ and $\gamma \in \mathbb{C}$ such that*

$$f = (g + \overline{h} + \gamma)|_A.$$

Proof. There are $G, H \in H_0^2$ and $\gamma \in \mathbb{C}$ such that

$$f = G + \overline{H} + \gamma.$$

Next, $g = PG$, $h = PH \in H_0^2(A, \mathcal{C})$ and a.s. on A,

$$(g + \overline{h} + \gamma) = (PG + \overline{PH} + \gamma) = \mathbb{E}_{m_A}^{\mathcal{C}}(f) = f.$$

CLAIM 3. *Let $A \in \mathcal{B}(\mathbb{T})$ and let $\mathcal{C}, \mathcal{C}' \subseteq \mathcal{B}(A)$ be such that both $(A, \mathcal{C})$ and $(A, \mathcal{C}')$ are analytic pairs. Then*

$$\mathcal{C} \le \mathcal{C}' \iff H_0^2(A, \mathcal{C}) \subset H_0^2(A, \mathcal{C}').$$

Proof. $\Rightarrow$ follows from the definition and $\Leftarrow$ follows from Claim 2.

([7]) See also [Sak07, Thm. 5.6].

CLAIM 4. $L := \overline{\mathtt{span}}\{fg : f \in H_0^2(A,\mathcal{C}),\ g \in H_0^\infty(A,\mathcal{C})\} \subsetneq H_0^2(A,\mathcal{C})$.

Proof. Let

$$d := \min\{k \ge 0 : \chi^{-k} g \in H^2\ \forall g \in H_0^2(A,\mathcal{C})\}.$$

Then $d \in \mathbb{N}$.

By Claim 2, there exists nonconstant $G \in H_0^\infty(A,\mathcal{C})$ with $\chi^{-1}G \in H^2$. Thus $\chi^{-d-1}g \in H^2$ for all $g \in L$, whence $H_0^2(A,\mathcal{C}) \setminus L \neq \emptyset$.

CLAIM 5. *Every $\phi \in H_0^2(A,\mathcal{C}) \cap L^\perp$ with $\|\phi\|_2 = 1$ is inner with $\phi(0) = 0$.*

Proof. For $n \ge 1$, $P(\phi\chi^n) \overset{\text{Claim 1}}{=} \phi P(\chi^n) \in L$, whence $\phi \perp \phi P(\chi^n)$ ($\because\ \phi \in L^\perp$). Thus for $n \ge 1$,

$$\widehat{(|\phi|^2)}(n) = \langle \phi, \phi\chi^n = \langle \phi, P(\phi\chi^n)\rangle = \langle \phi, \phi P(\chi^n)\rangle = 0,$$

whence also $\widehat{(|\phi|^2)}(-n) = \overline{\widehat{(|\phi|^2)}(n)} = 0$ and

$$|\phi|^2 = \widehat{(|\phi|^2)}(0) = \|\phi\|_2^2 = 1. \ \blacksquare$$

Fix $\phi \in H_0^2(A,\mathcal{C}) \cap L^\perp$ (inner with $\phi(0) = 0$).

CLAIM 6. $H_0^2(A,\mathcal{C}) = H_0^2 \circ \tau = H_0^2(\mathbb{T}, \tau^{-1}\mathcal{B})$, *where* $\tau = \tau(\phi)$.

Proof. Since $\phi \in H_0^\infty(A,\mathcal{C})$, we have (i) $A \cap \tau^{-1}\mathcal{B}(\mathbb{T}) \subseteq \mathcal{C}$ and (ii) $\phi^n \in H_0^\infty(A,\mathcal{C})$ for all $n \ge 1$. It follows that every $F \in H_0^2 \circ \tau$ is in $H_0^2(A,\mathcal{C})$ having the form $F \circ \chi = \sum_{n\ge1} a_n\phi^n$ with $(a_k : k \ge 1) \in \ell^2$. Thus $H_0^2(A,\mathcal{C}) \supseteq H_0^2 \circ \tau$.

To show equality we will prove that

$$M := H_0^2(A,\mathcal{C}) \cap (H_0^2 \circ \tau)^\perp = \{0\}.$$

To this end, we show first that

$$\overline{\phi} M \subset M. \tag{39}$$

Indeed, let $g \in M$; then $g \perp \phi^j$ for all $j \ge 1$ and

$$\langle \overline{\phi} g, \phi^j\rangle = 0 \quad \forall j \ge 0. \tag{40}$$

For $k \ge 1$,

$$\langle \overline{\phi} g, \chi^{-k}\rangle = \langle g\chi^k, \phi\rangle \overset{(38)}{=} \langle P(g\chi^k), \phi\rangle = \langle gP(\chi^k), \phi\rangle = 0.$$

Thus $\overline{\phi} g \in H_0^2(A,\mathcal{C})$.

Each $H \in H_0^2(A, \tau^{-1}\mathcal{B})$ is of the form $H = h \circ \tau$ with $h \in H_0^2$. By (40),

$$\overline{\phi} g \perp \sum_{k\ge0} \hat{h}(k)\phi^k = h \circ \tau = H$$

and $\overline{\phi} g \in M$, proving (39).

To see that $M = \{0\}$, suppose otherwise: let $g \in M$, $g \not\equiv 0$; then by (39) (used repeatedly) $\overline{\phi}^j g \in M$ for all $j \ge 1$, which is impossible unless $g \equiv 0$. $\blacksquare$

Claim 7. $A \in \tau^{-1}\mathcal{B}(\mathbb{T})$.

Proof. By Claim 6, $H_0^2(A, \mathcal{C}) = H_0^2 \circ \tau$ and $Pf = \widehat{\tau}(f) \circ \tau$ and

$$\mathbb{E}_{m_A}^{\mathcal{C}}(f) = P(1_A f)|_A$$

for all $f \in H_0^2$ and hence $f \in L^2$.

In particular, $(\widehat{\tau}(1_A) \circ \tau)|_A = 1$ and we claim that $\widehat{\tau}(1_A) \circ \tau = 1_A$.

To see this, note that

$$\widehat{\tau}(1_A) \circ \tau = 1_A + J,$$

where $J := \widehat{\tau}(1_A) \circ \tau \cdot 1_{A^c} \geq 0$.

Now,

$$m(A) = \mathbb{E}_m(\widehat{\tau}(1_A) \circ \tau) = m(A) + \mathbb{E}_m(J),$$

whence $\mathbb{E}_m(J) = 0$, $J = 0$ a.s., $\widehat{\tau}(1_A) \circ \tau = 1_A$ and $A \in \tau^{-1}\mathcal{B}(\mathbb{T})$. ∎

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