AN INTRODUCTION TO INFINITE ERGODIC THEORY NOTES AND CORRECTIONS AUGUST 2026

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¶`Updates in §3.2 & §3.8`

§1.0. `page 9, Insert before line -8:`

The function $p : X \to \mathbb{R}$ defined by $p(x) := m_{\pi(x)}(\{x\})$ is measurable. To see this let note that if $\alpha \subset \mathcal{B}$ is a countable partition of X, then $x \mapsto m_{\pi(x)}(\alpha(x))$ is measurable where $x \in \alpha(x) \in \alpha$. If α_n is a refining sequence of countable measurable partitions of X such that $\underset{a\in\alpha_n}{\to} \sup \text{diam.}(a) \to 0$ as $n \to \infty$ then $m_{\pi(x)}(\alpha_n(x)) \to p(x)$ a.e., whence p is measurable.

`page 10, line -1 should read`

isomorphic if there is an isomorphism between them.

`page 13, lines 1 through 6 should read`

we have that $\psi \circ \phi(x) \le x \ \forall \ x \in X$. Moreover, $m \circ \phi^{-1} = \mu \times \lambda$ where λ is Lebesgue measure on $[0,1]$ and $\mu \times \lambda \circ \psi^{-1} = m$. It follows that $m \circ (\psi \circ \phi)^{-1} = m$ whence $\psi \circ \phi(x) = x$ for a.e. $x \in X$ and $\phi : X \to Y \times [0,1]$ is a measure space isomorphism.

`page 13 line -4 to page 14 line 4 should read`

$\mathfrak{B}(L^2(\nu))$ can be equipped with the *strong topology*, defined by the metric

$$\rho(Q,R) := \sum_{n=1}^{\infty} \frac{1}{2^n}(\|Qf_n - Rf_n\|_2 + \|Q^{-1}f_n - R^{-1}f_n\|_2)$$

where $\{f_n : n \in \mathbb{N}\}$ is a orthonormal basis in $L^2(\nu)$, but it is neither a Polish space nor a topological group (under composition).

The subgroup of invertible unitary operators (isometries) $\mathcal{U}(L^2(\nu))$ forms a Polish topological group.

§1.1. `page 21, line 5 should read`

By assumption $a_n \uparrow \infty$ as $n \uparrow \infty$, whence $\phi_n \to 0$ as $n \to \infty$ on $A \setminus A_\infty$.

`page 28, lines -3, -4 should read`

$$P_1 \in \mathscr{P}(Y^{\mathbb{N}}) \ \ni \ P_1([A_1, \cdots, A_n]) = P_n(A_1 \times \cdots \times A_n),$$

$$P_2 \in \mathscr{P}(Y^{\mathbb{Z}}) \ni P_2([A_1,\cdots,A_n]_k) = P_n(A_1\times\cdots\times A_n),$$

§1.4. `page 37, line -11`
$\|\widehat{T}f\|_p \le M$ `should be` $\|\widehat{T}^n f\|_p \le M \ \forall\ n \ge 0$

`page 40, line -16 should read`
and $m(B_1 \cap T^{-(n_2-n_1)}B_2) = \int_{B_2} \widehat{T}^{n_2-n_1} 1_{B_1} dm < \frac{\varepsilon}{2^2}$.

`page 40, lines -12, -11 should read`

$$\sum_{j=1}^{k-1} m(B_j \cap T^{-(n_k-n_j)}B_k) = \int_{B_k} \widehat{T}^{n_k-n_{k-1}}(\sum_{j=1}^{k-1}\widehat{T}^{n_{k-1}-n_j}1_{B_j})dm < \frac{\varepsilon}{2^k} \quad (k\ge 1).$$

It follows that $W := \bigcap_{k=1}^{\infty} B_k \setminus \bigcap_{1\le i<j<\infty} B_i \cap T^{-(n_j-n_i)}B_j \in \mathfrak{W}$ and $m(W) > m(\mathfrak{N}) - 2\varepsilon$. $\square$

§1.5. `page 44, line 1 should read`

Proof We prove the lemma for $B \in \mathcal{B}$ of finite measure. The general case follows by monotonicity. For $B \in \mathcal{B},\ m(B) < \infty$, define for $n \ge 0$ .....

§1.6. `page 51, line -1 should read`

Define maps $L,\ R : G \to \mathcal{M}(G)$, the measure multiplying transformations of $(G, \mathcal{B}(G), m_G)$, by $L_g(x) := gx,\ R_g(x) := xg$.

§2.2. `on page 57, line -1 and page 58, lines 2, 6 and 8,`
η_n `should be` η_{n+1}.

`page 59, lines 14-20 should read`,

Let $f \in L^1(m)$. Fix $\varepsilon > 0$. We can write $f = g + k$, where $\|k\|_1 < \varepsilon^2$. It follows that

$$\limsup_{n\to\infty} |R_n(f,p) - \Phi_p(f)| \le \limsup_{n\to\infty} |R_n(k,p) - \Phi_p(k)| \le \sup_{n\in\mathbb{N}} |R_n(k,p)| + |\Phi_p(k)|,$$

whence, by the maximal inequality,

$$m_p([\overline{\lim_{n\to\infty}} |R_n(f,p) - \Phi_p(f)| > 2\varepsilon]) \le m_p([|\Phi_p(k)| > \varepsilon]) + m_p([\sup_{n\in\mathbb{N}} |R_n(k,p)| > \varepsilon])$$
$$\le \frac{2\|k\|_1}{\varepsilon} \le 2\varepsilon.$$

This last inequality holds for arbitrary $\varepsilon > 0$, whence

$$\limsup_{n\to\infty} |R_n(f,p) - \Phi_p(f)| = 0 \ \text{ a.e.},$$

§2.6. `page 74, line 14 should read`

$$\frac{1}{n}\sum_{k=0}^{n-1}|m(A\cap T^{-n}B)-m(A)m(B)|\to 0 \quad \forall \quad A,B\in\mathcal{B};$$

`page 76, line 19 should read`

$$\exists\ \psi : e(T)\times X_T\to S^1 \text{ jointly measurable, such that}$$
$$\psi(t,Tx)=e^{2\pi it}\psi(t,x)\ m_T-\text{a.e. } \forall\ t\in e(T).$$

`throughout pages 78 and 79,`
γ_k `should be` $\gamma(k)$

`page 80, line 2 should read`

$$\Phi_n(x):=s\sum_{k=1}^{n}\gamma(k)x_k=\sum_{k=1}^{n}x_k\epsilon_k\langle\gamma(k)s\rangle \qquad \text{mod } 1.$$

`page 80, line 5 should read`

$$\Phi_n(x):=s\sum_{k=1}^{n}\gamma(k)x_k=\sum_{k=1}^{n}x_k(\epsilon_k\langle\gamma(k)s\rangle+\nu_k)=\sum_{k=1}^{n}x_k\epsilon_k\langle\gamma(k)s\rangle \qquad \text{mod } 1.$$

`page 80, in lines 6,9,15,17,19,24`
`delete` γ_k

`page 80, in line 11 and page 81 in line 8`
γ_k `should be` $\gamma(k)$

§3.2. `page 93, lines -15, -14 should read`

The group $\mathfrak{A}_1(X,\mathcal{B},m)$ of invertible, measure multiplying transformations of $(X,\mathcal{B},m)$ as a closed subgroup of the unitary operators on $(X,\mathcal{B},m)$ is a Polish group.

`page 94, lines 1,2 should read`

It follows that if $Q : T \overset{c}{\leftrightarrow} T$, then $Q(x,n)=(qx,n+\psi(x))$ where $q : W\to W$ is an invertible nonsingular map with $\mu\circ q^{-1}=c\mu$ and $\psi : W\to\mathbb{Z}$ is measurable, whence

`page 98, the remark should read`

Remark

It was shown in [Aar87] that $\exists\ L : \{0,1\}\to[0,\infty)$ such that for every conservative, ergodic measure preserving transformation T, $\exists\ c_T,\ 0< c_T<\infty$ such that

$$L(1_A,1_A\circ T,\dots)=c_T m_T(A) \quad \text{mod } \Delta(T)\ m_T-\text{ a.e. } \forall\ A\in\mathcal{B}_T,\ m_T(A)<\infty.$$

Thus if $\Delta(T) = \{1\}$, then

$$L(1_A, 1_A \circ T, \dots) = c_T m_T(A) \ \ m_T - \text{ a.e. } \forall\ A \in \mathcal{B}_T,\ \ m_T(A) < \infty.$$

This does not entail existence of a law of large numbers for T. A suitable example is given in chapter 8 (in view of which it is seen that the definition of "a law of large numbers for T" given in [Aar87] is different from the one here).

§3.4. `page 102, line -17 should read`
(see chapter 8 for more on skew products).

§3.5. `page 109, line 5 should read`
Call $R \in \mathfrak{A}_0$ *n-cyclic* if R^n =Id, and $\exists$ a partition $\{A_1, \dots, A_n\} \subset \mathcal{B}$

`page 109, line 13 should read`

$$Rx = \begin{cases} Tx & x \in \bigcup_{k=0}^{n-2} T^k E, \\ T^{-(n-1)}x & x \in T^{n-1}E, \\ R_k x & x \in E_k \ \ (1 \le k \le n) \end{cases}$$

`page 109, line -8 should read`
Given $\epsilon > 0$, $\exists$ a partition $\alpha \subset \mathcal{B}$ with $m(a) = c\ \forall\ a \in \alpha$ and subsets

`page 111, line 15, delete` by step 1,

`page 111, line -15 should read`
The lemma is now established by ergodicity of T and Hopf's theorem.

§3.6. `page 113, line 9 should read`
Suppose that $n_k,\ d_k \to \infty$, then $\exists\ m_\ell := n_{k_\ell} \to \infty$ and a random

`page 114, lines -6 to -3 should read`

Proof Choose $A \in \mathcal{B}$, $m(A) = 1$. In case $d_k \to \infty$, by proposition 3.6.1, and positivity $\exists\ m_\ell := n_{k_\ell} \to \infty$, and a random variable Y on $[0, \infty]$ such that

$$\frac{S^T_{m_\ell}(1_A)}{d_{k_\ell}} \xrightarrow{\mathfrak{L}} Y.$$

In case d_k is bounded, $\frac{S_{n_k}(1_A)}{d_k} \xrightarrow{\mathfrak{L}} \infty$. The result follows from Hopf's theorem. □

§3.7. `page 121, line 2 should read`

$$\sum_{n=0}^{\infty} e^{-\lambda n} \left| \int_A S_n(1_A)^p dm - p! \int_A a(p,n) dm \right| \le \sum_{q=1}^{p-1} \gamma_p(q) \sum_{n=0}^{\infty} e^{-\lambda n} \int_A a(q,n) dm$$

`page 122, line 1 should read`
Putting it all together, we obtain that for $\lambda < \lambda_{p-1}$,
`page 122, line -1 should read`

$$a(p,k)^2 \le S_k(1_A)^{2p} = \sum_{q=1}^{2p} \gamma_{2p}(q) a(q,k) \le M_p a(2p,k),$$

`page 126, line -4, and page 127, line 4 should read`

$$\sum_{n=0}^{N} \widehat{T}^n 1_B = \sum_{k=0}^{N} \widehat{T}^k (1_{A_k} \sum_{n=0}^{N-k} \widehat{T}^n 1_B) + \sum_{n=0}^{N} \widehat{T}^n 1_{B \setminus \bigcup_{j=0}^{n} T^{-j} A}.$$

`page 127, line -2 should read`
Let $M_k \downarrow 1$ be such that
`page 128, line 5 should read`

$$\le m(A) a(N) \sum_{k=0}^{N} M_{N-k} \widehat{T}^k 1_{A'_k}$$

`page 128, line -1 should read`

$$\frac{1}{a(n)} \sum_{k=0}^{n-1} \widehat{T}^k 1_B \le M \text{ a.e. on } X \ \forall \ n \ge 1.$$

§3.8. Theorem 3.8.3
The theorem is true, but the proof given contains an error. It does not follow (on p. 137, l.7) that

$$L_A(n) - L_B(n) = \sum_{k=0}^{n} (m(A_k) - m(B_k)) \le e(c_A(\tfrac{1}{n}) - c_B(\tfrac{1}{n})).$$

A correct proof is in [AZ14, §5] which uses Korenblyum's Tauberian theorem ([Kor55], see also [BGT87, Theorem 2.10.1]) to obtain minimal wandering rates from the the asymptotic renewal equation.

§4.2. `page 142, line 15 should read`

the subgroup generated by $K_x - K_x = \mathrm{Per}\,(x)\mathbb{Z}$.

§4.3. `page 143, line -7 should read`

$$f' := \frac{dm \circ f}{m} \equiv |Df|.$$

`page 147, lines 10, 11 should read`

... An indifferent fixed point $x_a \in a \in \alpha$ is a *regular source* if $DT \downarrow$ on $a_- := a \cap (-\infty, x_a)$, and $DT \uparrow$ on $a_+ := a \cap (x_a, \infty)$ strictly.

`page 148, line 10 should read`

$$|D^2 v_g| \le C|Dv_g| \text{ on } \mathscr{D}(v_g) \ \forall \ \ g \in \alpha^* \qquad (4).3.1$$

§4.4. `page 152, line 8 should read`

In particular, $\mathfrak{C}$*, and* $\mathfrak{D}$ *are both unions of sets in* $\mathfrak{r}$*.*

§4.7. `page 165, lines 7-8 should read`

4) The collection of Lipschitz continuous functions on X is denoted by L and equipped with the norm $\|f\|_L := \|f\|_{L^1(m)} + D_X f$.

`page 165, lines -11 - -7 should read`

We'll call a pair of Banach spaces $(\mathcal{C}, \mathcal{L})$ *adapted* if $\mathcal{L} \subset \mathcal{C}$, $\|\cdot\|_{\mathcal{C}} \le \|\cdot\|_{\mathcal{L}}$, $(\overline{\mathcal{L}})_{\mathcal{C}} = \mathcal{C}$,

$$x_n \in \mathcal{L}\ (n \ge 1),\ \sup_n \|x_n\|_{\mathcal{L}} < \infty,\ x_n \xrightarrow{\mathcal{C}} x \implies x \in \mathcal{L}, \|x\|_{\mathcal{L}} \le \sup_n \|x_n\|_{\mathcal{L}},$$

and $\mathcal{L}$-bounded sets are precompact in $\mathcal{C}$.

`page 166, lines -15, -12, -9 and -8`

f_n should be v_n.

§4.8. `page 172, lines -8,-7 should read`:

It follows from theorem 4.8.3 (below) that T has minimal wandering rates in the sense that

`page 173, line -6`:

by (4.8.1) `should read` by (4.3.2)

`page 178, line -7 should read`:

We first prove the lemma for slowly varying L with constant k. In this case, it follows that

`page 179, line 5 should read`:

An arbitrary slowly varying L is asymptotically approximated by one with constant k, and the general case of the lemma follows from a standard monotonicity argument. □

`page 180, line 14 should read`:

Next, $B \in \mathcal{U}(T)$ (being a Darling-Kac set). By theorem 4.8.3, $L(n) \sim L_B(n)$.

§5.2. `page 187, line -19 should read`

Since $S\times T_u$ is the natural extension of $S\times T$, we have that $S\times T_u$ is

`page 187, lines -6, -5 should read`

3) Show using the Darling-Kac theorem that for $\beta\in(0,1)$, $\exists\ c_\beta\in\mathbb{R}_+$ such that $E(e^{-\frac{t}{Y_\beta^\beta}})=e^{-c_\beta t^\beta}$ where Y_β has the Mittag-Leffler distribution of order β.

`page 187, in line -2`

α `should be` $1-\alpha$.

§6.1. `page 203, in lines 2, 3, and 5,`

dm should be dy.

`page 203, in line -3,`

$\varphi\circ g$ should be $\varphi\circ\gamma$.

`page 205, line 14.  Change` 6.1.4 `to` 6.1.5

`page 206, lines -10, -9 should read`

Proof. For $z\in U$ by proposition 6.1.1,

$$\widehat{T}^n p_z(x)=\frac{1-|f^n(z)|^2}{|e^{2\pi ix}-f^n(z)|^2}\to 0 \text{ as } n\to\infty$$

`page 207, line 7 should read`

$$\mathfrak{b}_f(z):=\sum_{n=1}^{\infty}1-|f^n(z)|.$$

§6.2. `page 210, line 2 should read`

$$v(z)=\alpha_T\mathrm{Im}\,z+\int_0^1\mathrm{Im}\,\frac{1+\tan\pi xz}{\tan\pi x-z}d\mu(x)=\alpha_T\mathrm{Im}\,z+\int_{\mathbb{R}}\mathrm{Im}\,\frac{1+tz}{t-z}d\mu_T(t)$$

`page 210, line 22.  Add`

Remark The formula $P_\omega\circ T^{-1}=P_{T(\omega)}$ was established in [Boo57] for inner functions of form $Tx=\alpha x+\beta+\sum_{k=1}^{\infty}\frac{p_k}{t_k-x}$.

`page 211, lines 2,3 should read`

Proof. Choose $\beta\in\partial U$ and set $f:=\phi_\beta^{-1}\circ T\circ\phi_\beta$, then f has Denjoy-Wolff point β iff $\alpha_T\geq 1$; this following from (1) in the Denjoy-Wolff theorem.

`page 211, line 6 should read`

If $T:\mathbb{R}^{2+}\to\mathbb{R}^{2+}$ is an inner function and $\alpha_T>0$, then, $m\circ T^{-1}=\frac{1}{\alpha_T}m$.

`page 211, line 10 should read`

then $\frac{\mathrm{Im}\,T(ib)}{b}\to\alpha_T$ and $\frac{\mathrm{Re}\,T(ib)}{b}\to 0$ as $b\to\infty$ whence

§6.3. `page 212, lines -6 to -1 should read`
To see this, note first that

$$\operatorname{Im} T^n(z) \uparrow \text{ as } n \uparrow,$$

whence

$$\operatorname{Im} G(T^n z) = \lim_{k\to\infty} \frac{T^{n+k}(x) - a_k}{b_k} \uparrow \text{ as } n \uparrow .$$

In particular,

$$\operatorname{Im} A^n(i) = \operatorname{Im} A^n \circ G(i) = \operatorname{Im} G \circ T^n(i) \uparrow \text{ as } n \uparrow .$$

§6.4. `page 218. Delete line -6`:

$$\le 1 + \frac{\mu(\mathbb{R})}{b_n}$$

`and add after the formula`: The convergence $\int_{\mathbb{R}} \left(\frac{1+t^2}{t^2+b_n^2}\right) d\mu(t) \to 0$ as $n \to \infty$ is established by the bounded convergence theorem.

`page 220, line -10 insert`

b) We perform the next estimation for L with constant k. The general case will follow by a standard monotonicity argument (c.f. the (corrected) proof of lemma 4.8.6).

§7.4. `page 233, line 13.`
$\forall\ s \in \mathbb{R}$ should be $\forall\ s \neq 0$.

§7.5. `page 235, line -8 should read`

$$a_\Gamma(x, y; J) := \sum_{\gamma\in\Gamma,\ \rho(x,\gamma y)\in J} e^{-\rho(x,\gamma y)} \text{ for intervals } J \subset [0, \infty).$$

`page 236, line -2 should read`

$$= \sum_{\gamma\in\Gamma} \int_{\mathbb{T}} \int_{\tanh(\frac{u}{2})}^{\tanh(\frac{v}{2})} 1_{\varphi_z^{-1}\gamma N_\rho(y,\epsilon)}(re^{2\pi i\theta}) \frac{dr d\theta}{1 - r^2}$$

`page 240, line -9 should read`

Since $N_\rho(y, \epsilon)$ is the Euclidean ball with centre $\frac{(1-\delta^2)y}{1-\delta^2|y|^2}$ and radius $\frac{\delta(1-|y|^2)}{1-\delta^2|y|^2}$

`page 243, line 5 should read`

$$m_\Gamma((\Delta(x, \epsilon))^2 a(t) \sim \int_0^t m_\Gamma(\Delta(x, \epsilon) \cap \varphi_\Gamma^{-s} \Delta(y, \epsilon)) ds$$

§8.1. `page 249, lines 1-5 should read`

REMARK The "only if" part of proposition 8.1.2 can be generalised to the case where T is a conservative, ergodic non-singular transformation (see theorem 5.5 in [Sch77]).

The "if" part may fail in case T is a conservative, ergodic measure preserving transformation of an infinite measure space.

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