DUAL MIXING AND QUASI-MIXING OF INNER FUNCTION RESTRICTIONS

JON AARONSON

Abstract. Dual quasi-mixing and dual mixing of a measure preserving transformation are uniform individual ratio limit properties of the transfer operator which entail the quasi-mixing and mixing properties of Krickeberg (respectively). The real restriction of a normalised, parabolic inner function of the upper half plane preserves Lebesgue measure and is dual mixing iff it is exact. Certain other restrictions are "singular" dual quasi-mixing.

Contents

§1 Introduction

§1.1 Overview.

Motivated by an example of E.Hopf ([Hop37, §17]), Krickeberg ([Kri67, Kri69]) defined notions of topological mixing and quasi-mixing for infinite measure preserving transformations (called "Hopf-Krickeberg mixing/quasi-mixing" in the sequel).

2020 *Mathematics Subject Classification.* 37A40,60J05,30J05,37A25.

Key words and phrases. measure preserving transformation, transfer operator, ratio limit property, mixing, quasi-mixing, inner function restriction.

A stronger, transfer operator version of Hopf-Krickeberg mixing (which we call "dual mixing") was established for certain "intermittent" interval maps in [Tha00].

This paper considers "dual mixing" and "dual quasi-mixing" of "normalised, parabolic inner function restrictions" which, in particular, provide new examples of Hopf-Krickeberg quasi-mixing transformations.

We complete this overview with two illustrative examples, both real restrictions of rational, parabolic, inner functions of the upper half plane (as in §1.5).

Let $(\mathbb{R}, m)$ be the real line equipped with Lebesgue measure.

Example 1. Let $T : \mathbb{R} \circlearrowleft$ be *Boole's transformation* as in [AW73]:

$$T(x) := x - \tfrac{1}{x}$$

and set $u_n = \frac{1}{\pi\sqrt{2n}}$, then $(\mathbb{R}, m, T)$ is a measure preserving transformation and

$$☕ \qquad \tfrac{1}{u_n}\widehat{T}^n f \xrightarrow[n\to\infty]{\text{u.c.s.}} \int_{\mathbb{R}} f dm \;\forall\; f \in C_C(\mathbb{R})$$

where $C_C(\mathbb{R}) := \{\texttt{continuous functions on } \mathbb{R} \texttt{ with compact support}\}$.

Here $\widehat{T} : L^1(m) \circlearrowleft$ is the *transfer operator* as defined by 🐾 on p.2 and u.c.s means "uniformly on compact subsets".

In other words, $(\mathbb{R}, m, T)$ is `dual mixing` as defined by ⚔ (on p.2).

Example 2.

Define $\tau : \mathbb{R} \circlearrowleft$ by

$$\tau(x) := T(x) + 1 = x - \tfrac{1}{x} + 1,$$

then $(\mathbb{R}, m, \tau)$ is a measure preserving transformation and there is a singular, Borel measure $\rho \not\equiv 0$ on $\mathbb{R}$ which is locally finite (in the sense that compact sets have finite ρ-measure) so that with $v_n = \frac{1}{n^2}$,

$$⚽ \qquad \tfrac{1}{v_n}\widehat{\tau}^n f \xrightarrow[n\to\infty]{\text{u.c.s.}} \int_{\mathbb{R}} f d\rho \;\forall\; f \in C_C(\mathbb{R}).$$

In other words, $(\mathbb{R}, m, T)$ is `singular dual quasi-mixing` as defined by 🕮 (below).

Proposition 3.1 on p.11 establishes ☕ and ⚽ .

§1.2 Dual mixing and quasi-mixing.

Let X be a locally compact, Polish space, let m be a locally finite, Borel measure on X and let (X, m, T) be a measure preserving transformation with *transfer operator* $\widehat{T} : L^1(m) \circlearrowleft$ defined by

$$🐾 \qquad \int_X \widehat{T} g \cdot h dm = \int_X g \cdot h \circ T dm \text{ for } g \in L^1(m),\; h \in L^\infty(m).$$

Motivated by [Kri67], [Ore71, Chapter 3], and [Tha00], we'll call (X, m, T) *dual mixing* if $\exists\; u_n > 0$ so that

$$⚔ \qquad \tfrac{1}{u_n}\widehat{T}^n f(x) \xrightarrow[n\to\infty]{\text{u.c.s}} m(f) := \int_X f dm \;\forall\; f \in C_C(\mathbb{R}).$$

.

More generally, motivated by [Kri67, Kri69, Pap67], for $\mu \not\equiv 0$ a locally finite, Borel measure on X, we'll call (X, m, T) *μ-dual quasi-mixing* if $\exists\ u_n > 0$ so that

$$\frac{1}{u_n}\widehat{T}^n f(x) \xrightarrow[n\to\infty]{\text{u.c.s}} \mu(f) := \int_X f d\mu \ \ \forall\ f \in C_C(\mathbb{R}). \tag{🕮}$$

In this case, (X, m, T) is *dual mixing* if $\mu = m$; and we'll call (X, m, T) *singular dual quasi-mixing* if $\mu \perp m$.

Remark

It is standard to extend 🕮 to the class of μ-integrable functions with compact support which are continuous at μ-a.e. point of $\mathbb{R}$, whence if (X, m, T) is μ-dual quasi-mixing, then

$$m(A \cap T^{-n}B) = \int_B \widehat{T}^n 1_A dm \sim u_n m(B)\mu(A) \text{ as } n \to \infty$$
$$\forall\ A \text{ bounded},\ \mu(\partial A) = 0\ \&\ B \in \mathcal{B}(X),\ B \text{ bounded}.$$

In particular, (X, m, T) is (μ, m)-Krickeberg quasi-mixing as in ❦ below.

Dual mixing for certain intermittent interval maps (including Boole's transformation as in ☕) was established in [Tha00]. For more examples, see [MT12].

§1.3 Hopf-Krickeberg mixing and quasi-mixing.

Let X be a locally compact, Polish space and let m be a locally finite, Borel measure on X.

A measure preserving transformation (X, m, T) is called
Hopf-Krickeberg (μ_1, μ_2)-quasi-mixing (abbr. HK-(μ_1, μ_2)-quasi-mixing)
(where $\mu_1, \mu_2 \not\equiv 0$ are locally finite, Borel measures on X) if $\exists\ u_n > 0$ so that

$$\frac{m(A \cap T^{-n}B)}{u_n} \xrightarrow[n\to\infty]{} \mu_1(A)\mu_2(B)\ \forall\ A \in \mathcal{R}_{\mu_1},\ B \in \mathcal{R}_{\mu_2} \tag{❦}$$

where $\mathcal{R}_\mu := \{A \subset X :\ \mu(\partial A) = 0\}$; and (X, m, T) is called *Hopf-Krickeberg mixing* (abbr. HK-mixing) if (X, m, T) is HK-(m, m)-quasi-mixing.

For Markov shifts, HK-mixing is characterised by Orey's `strong ratio limit property` (abbr. `SRLP`) as in [Ore61]. See [Kri67, Kri69, Pap67].

In addition, a variety of smooth dynamical systems are now known to be HK-mixing. See e.g. [Len10, DN22, FR24, PT25].

A characterisation of HK-quasi-mixing Markov shifts with discrete state spaces in terms of Pruitt's generalized `SRLP` (as in [Pru65]) is given by [Pap67, Theorem 2.4]. Examples of HK-quasi-mixing Markov shifts with continuous state space can also be found in [Pap67].

§1.4 Ergodic properties.

Dual quasi-mixing transformations.

A dual quasi-mixing measure preserving transformation (X, m, T) of a probability space is *exact* in the sense that $\bigcap_{n\ge 0} T^{-n}\mathcal{B}(X) \overset{m}{=} \{\varnothing, X\}$.

By Hopf's decomposition results (see e.g. [Aar97, Proposition 1.3.1]), if (X, m, T) is μ-dual quasi-mixing, then (X, m, T) is conservative iff $\sum_{n\ge 1} u_n = \infty$.

In this case, by the ratio ergodic theorem, almost every `ergodic component` (X, m_ω, T) (of the ergodic decomposition of (X, m, T) as in e.g. [Aar97, 2.2.9]) is `pointwise dual ergodic` (as in [Aar97, §3.7]) with $m_\omega = \mu$, whence $\mu = m$ and (X, m, T) is dual mixing which in turn entails `conditional rational weak mixing`. See [AT20].

HK-quasi-mixing transformations.

As shown in [Kri67, Kri69, Pap67], a probability preserving transformation is HK-quasi-mixing iff it is strongly mixing.

A `rationally ergodic` measure preserving transformation (X, m, T) (see e.g. [Aar97, §3.3]) which is (μ_1, μ_2)-HK-quasi-mixing as in ❦ on p. 3 with $\mathcal{R}_{\mu_i} \cap R(T) \neq \varnothing$ $(i = 1, 2)$ is necessarily HK-mixing and hence by [Aar13, Lemma C] `rationally weakly mixing`.

However, we do not know whether a general, conservative, HK-mixing transformation preserving an infinite measure is necessarily ergodic.

§1.5 Inner functions of the upper half plane.

An analytic endomorphism $T : \mathbb{R}^{2+} \circlearrowleft$ is called an *inner function* of the *upper half plane* $\mathbb{R}^{2+} := \{x + iy : \ x, y \in \mathbb{R}, \ y > 0\}$ if

$$\text{for } m\text{-a.e. } x \in \mathbb{R}, \ \exists \lim_{y \to 0+} T(x + iy) =: T(x) \in \mathbb{R}.$$

An inner function $T : \mathbb{R}^{2+} \circlearrowleft$ is necessarily of form

🚗
$$T(z) = \alpha_T z + \beta_T + \int_{\mathbb{R}} \tfrac{1+tz}{t-z} d\mu_T(t) \quad (z \in \mathbb{R}^{2+}).$$

where $\alpha_T \geq 0$, $\beta_T \in \mathbb{R}$ and μ_T is a singular measure on $\mathbb{R}$ (i.e. $\mu_T \perp m$).

The measurable mapping $T : \mathbb{R} \circlearrowleft$ is aka the *real restriction* of the inner function.

For $\omega \in \mathbb{R}^{2+}$, define $\varphi_\omega(t) := \mathtt{Im}\, \frac{1}{\pi(t-\omega)}$ & $P_\omega \in (\mathbb{R})$, $dP_\omega := \varphi_\omega dm$.

Theorem 1.1. Nordgren's Theorem [Nor68]

For $T : \mathbb{R}^{2+} \circlearrowleft$ inner, as in 🚗, $(\mathbb{R}, m, T)$ is a nonsingular transformation with $\widehat{T}\varphi_\omega = \varphi_{T\omega}$; equivalently

🦇
$$P_\omega \circ T^{-1} = P_{T(\omega)} \ \forall \ \omega \in \mathbb{R}^{2+}.$$

Remarks

- 🦇 is aka `Boole's formula` as a version for rational inner functions of the the upper half plane appears in [Boo57, p. 787] (see also [Gla77, §8]).
- For a converse to Nordgren's theorem, see [Let77].
- As a consequence of 🦇 (see [Let77]): if $\alpha_T > 0$, then

⚓
$$m \circ T^{-1} = \tfrac{1}{\alpha_T} m.$$

For $T : \mathbb{R}^{2+} \circlearrowleft$, we'll call the nonsingular transformation $(\mathbb{R}, m, T)$ the *restriction* of T.

Theorem 1.2. Denjoy-Wolff Theorem

Let $T : \mathbb{R}^{2+} \circlearrowleft$ be a non-Möbius inner function, then $\exists\ \mathfrak{d} = \mathfrak{d}_T \in \mathbb{R}^{2+} \cup \mathbb{R} \cup \{\infty\}$ so that

$$T^n z \xrightarrow[n \to \infty]{} \mathfrak{d} \ \forall \ z \in \mathbb{R}^{2+}.$$

- *If $\mathfrak{d} \in \mathbb{R}^{2+}$, then $T(\mathfrak{d}) = \mathfrak{d}$ and $|T'(\mathfrak{d})| < 1$.*
- *If $\mathfrak{d} = \infty$, then $\alpha_T \geq 1$.*
- *If $\mathfrak{d} \in \mathbb{R}$, then $\alpha_{T_\mathfrak{d}} \geq 1$ where $T_\mathfrak{d} := \phi_{\mathfrak{d}_T} \circ T \circ \phi_{-\mathfrak{d}_T}$ with $\phi_t(z) := \frac{1+tz}{t-z}$.*

Denjoy-Wolff points

The point $\mathfrak{d}_T$ is called the *Denjoy-Wolff point of* T.

The inner function T is called *elliptic* if $\mathfrak{d} \in \mathbb{R}^{2+}$; *parabolic* if $\mathfrak{d} \in \mathbb{R} \cup \{\infty\}$ and $\alpha_{T_\mathfrak{d}} = 1$ (with $T_\infty := T$); and *hyperbolic* if $\mathfrak{d} \in \mathbb{R} \cup \{\infty\}$ and $\alpha_{T_\mathfrak{d}} > 1$.

We'll also call a non-elliptic inner function T *normalised* if $\mathfrak{d}_T = \infty$. A non-elliptic inner function T is Möbius-conjugate to the normalised inner function $T_\mathfrak{d}$ as above.

If $T : \mathbb{R}^{2+} \circlearrowleft$ is parabolic and normalised, then by ⚓, (X, m, T) is a measure preserving transformation.

Theorem 1.3. Pommerenke's limit theorem [Pom79]

Suppose that $T : \mathbb{R}^{2+} \circlearrowleft$ *is normalised, inner and let* $T^n(i) = a_n + ib_n$, *then* $\exists\ \Pi : \mathbb{R}^{2+} \circlearrowleft$ *analytic, so that*

⛉ $$\frac{T^n - a_n}{b_n} \xrightarrow[n\to\infty]{u.c.s.} \Pi \ \textit{with}\ \Pi \circ T \equiv a\Pi + b \quad \textit{where}\ b \in \mathbb{R}, a \geq \alpha_T.$$

Moreover, if $T \neq \mathtt{Id}$, *then*

✎ $$\begin{aligned} \Pi \equiv i \quad &\Leftrightarrow\ \ \Pi \circ T \equiv \Pi \\ &\Leftrightarrow\ \ \langle T^n(z), T^n(\omega)\rangle \xrightarrow[n\to\infty]{} 0\ \forall\ z, \omega \in \mathbb{R}^{2+} \\ \textit{where}\ \ &\langle z, \omega\rangle := \left|\tfrac{z-\omega}{z-\overline{\omega}}\right|. \end{aligned}$$

We'll call the function $\Pi = \Pi_T : \mathbb{R}^{2+} \circlearrowleft$ the *Pommerenke limit of* T.

Hamilton showed ([Ham94, Theorem P]) that the Pommerenke limit is either constant or inner.

Since $V := \mathtt{Im}\,\Pi$ is positive, harmonic & $V(i) = 1$, $\exists\ \varpi_\Pi \in \mathscr{P}(\widehat{\mathbb{R}})$ so that

🔒 $$V(z) = \int_{\widehat{\mathbb{R}}} \frac{\varphi_z}{\varphi_i} d\varpi_\Pi = \varpi_\Pi(\{\infty\})\mathtt{Im} z + \int_{\mathbb{R}} \frac{\varphi_z}{\varphi_i} d\varpi_\Pi.$$

We'll call the measure ϖ_Π the *representing measure* of Π. Thus ([Ham94, Theorem P]) either $\varpi_\Pi = P_i$ (when $V \equiv i$) or $\varpi_\Pi \perp P_i$ (otherwise).

Note that $\mathtt{spt}\varpi_\Pi = \{\infty\}$ iff $\mathtt{Im}\,\Pi(z) = \mathtt{Im}\,z$ which is the case iff T is Möbius. Thus T non-Möbius entails $\varpi_\Pi(\mathbb{R}) > 0$.

§1.6 Ergodic theory of inner function restrictions.

In addition to 🦇 (p.4) also have

⊕ $$\int_{\mathbb{R}} |\varphi_z - \varphi_\omega| dm = \frac{4}{\pi} \sin^{-1}\langle z, \omega\rangle.$$

See e.g. [Aar97, Lemma 6.1.4].

By approximation, using ✈ (p.4); & ⊕, we have, for $T : \mathbb{R}^{2+} \circlearrowleft$ inner that

$$\begin{aligned} &\|\widehat{T}^n u\|_{L^1(m)} \xrightarrow[n\to\infty]{} 0\ \forall\ u \in L^1(m),\ \int_X u dm = 0 \iff \\ &\langle T^n z, T^n \omega\rangle \xrightarrow[n\to\infty]{} 0\ \forall\ z, \omega \in \mathbb{R}^{2+} \\ &\iff \text{by [Lin71], (see also e.g. [Aar97, Theorem 1.3.3])} \\ &\bigcap_{n\geq 0} T^{-n}\mathscr{B}(X) \overset{m}{=} \{\varnothing, \mathbb{R}\} \text{ i.e. } (\mathbb{R}, m, T) \text{ is exact.} \end{aligned}$$

By Pommerenke's theorem 1.3 (p.5), in case $(\mathbb{R}, m, T)$ is not exact, the Pommerenke limit is not constant and there is a non-constant T-invariant, bounded analytic function on $\mathbb{R}^{2+}$, whence $(\mathbb{R}, m, T)$ is not ergodic.

By Hopf's criterion (see e.g. [Aar97, Proposition 1.3.1]), $(\mathbb{R}, m, T)$ conservative iff $\sum_{n\ge 1} \widehat{T}^n \varphi_i = \infty$ a.e.; equivalently, $\sum_{n\ge 1} \frac{1}{b_n} = \infty$ with $b_n := \frac{|T^n(i)|^2}{\mathtt{Im}\, T^n(i)}$.

As in [Neu78] (see also [Aar78] & [Aar97, Proposition 6.1.7]), if a non-Möbius inner function is conservative, it is ergodic, hence exact.

However, as in [Aar81] (also [Aar97, Theorem 6.4.8]) there is a dissipative exact inner function restriction.

§1.7 Main Results.

Theorem M (`Mixing`)

Suppose that $T : \mathbb{R}^{2+} \circlearrowleft$ is a normalised, parabolic, inner function, then $(\mathbb{R}, m, T)$ is dual mixing iff it is exact.

Theorem Q (`Quasi-mixing`)

Suppose that $T : \mathbb{R}^{2+} \circlearrowleft$ is a normalised, parabolic, inner function; non-Möbius and not exact.

If $T^n(i) \to \infty$ tangentially, then $(\mathbb{R}, m, T)$ is singular dual quasi-mixing.

Here $T^n(i) = a_n + ib_n \in \mathbb{R}^{2+} \to \infty$ *tangentially* if $\Delta_n := (\frac{a_n}{b_n})^2 \xrightarrow[n\to\infty]{} \infty$.

§2 Dual mixing and quasi-mixing of inner function restrictions

In this section, we prove the main results.

§2.1 Uniform weak convergence of measures.

Let X be a locally compact, Polish space, and let

$$C_{\mathtt{lim}}(X) := \{f \in C(X) :\ \exists \lim_{x\to\infty} f(x) \in \mathbb{R}\}$$

equipped with the supremum norm.

Note that $C_{\mathtt{lim}}(X) \cong C(\widehat{X})$ where $\widehat{X}$ is the one-point compactification of X.

Let $\mathcal{M}(X)$ denote the space of finite Borel measures on X and let $\mathcal{P}(X)$ denote the Borel probabilities on X.

By Riesz's representation theorem, $\mathcal{M}(\widehat{X}) = C(\widehat{X})^*_+$ and by Helly's theorem $\mathcal{P}(\widehat{X})$ is weak * compact.

Let (Y, d) be a compact metric space. We'll call a function $\mu : Y \to \mathcal{M}(X)$ is *weakly continuous* if $x \mapsto \int_X f d\mu_x$ is continuous $(Y \to \mathbb{R})$ for each $f \in C_{\mathtt{lim}}(X)$. Denote

$$C(Y, \mathcal{M}(X)) := \{\mu : Y \to \mathcal{M}(X) :\ \mu \text{ weakly continuous}\}$$

considered with the topology of convergence

$$\mu_n \xrightarrow[n\to\infty]{C(Y,\mathcal{M}(X))} \nu \iff \sup_{x\in Y} |\mu_{n,x}(f) - \nu_x(f)| \xrightarrow[n\to\infty]{} 0\ \forall\ f \in C_{\mathtt{lim}}(X).$$

Proposition 2.1.

Let $(Y.d)$ be a compact metric space, then

$\mathcal{K} \subset C(Y, \mathcal{M}(\widehat{X}))$ is precompact iff for some $J \subset C_{\mathtt{lim}}(X)_+$ with dense linear span,

🍃 $$\omega_f(h) := \sup\{|\mu_x(f) - \mu_y(f)| : \ x, y \in Y, \ d(x,y) \le h, \ \mu \in \mathcal{K}\} \xrightarrow[h \to 0+]{} 0 \ \forall \ f \in J.$$

Proof Precompactness implies 🍃 by the Arzela-Ascoli theorem.

Conversely, suppose that $\mathcal{K}$ satisfies 🍃.

We claim first that $\mathcal{K}$ is *uniformly bounded* in the sense that

🌲 $$C := \sup\{\mu_x(\mathbb{1}) : \ \mu \in \mathcal{K}, \times \in Y\} < \infty \text{ where } \mathbb{1}(x) = 1 \ \forall \ x \in X.$$

If not, $\exists \ n_k \uparrow \infty, \ x_k \in Y$ so that $\mu_{n_k, x_k}(\mathbb{1}) \ge k$.

Fix $f \in J$ so that $f(x) \ge c > 0 \ \forall \ x \in \widehat{X}$.

By 🍃 and the Arzela-Ascoli theorem, $\exists \ k_\ell \to \infty$ & $g \in C(Y)$ so that

$$\varlimsup_{\ell \to \infty} \mu_{n_{k_\ell}, x_{k_\ell}}(f) \le C := \sup_{y \in Y} |g(y)|$$

entailing for large ℓ,

$$3C < k_\ell \le \mu_{n_{k_\ell}, x_{k_\ell}}(\mathbb{1}) \le 2\mu_{n_{k_\ell}, x_{k_\ell}}(f) \lesssim 2C. \ \boxtimes \ \text{☑} \ 🌲$$

To see precompactness of $\mathcal{K}$ fix an infinite subset $\mathcal{K}_0 \subset \mathcal{K}$.

By 🍃, $\exists \ \mu_k \in \mathcal{K}_0$ so that

$$\mu_{k,x}(f) \xrightarrow[k \to \infty]{\text{uniformly}} G(f)(x) \ \forall \ f \in \mathtt{Span}\, J$$

where $f \mapsto G(f)$ is positive, linear $\mathtt{Span}\, J \to C(Y)$.

Evidently $\|G(f)\|_{C(Y)} \le C\|f\|_{C_{\mathtt{lim}}(X)}$ with C as in 🌲 and so G extends by uniform approximation to a positive, linear map $G : C_{\mathtt{lim}}(X) \to C(Y)$ satisfying $\|G(f)\|_{C(Y)} \le C\|f\|_{C_{\mathtt{lim}}(X)}$.

By the Riesz representation theorem $\exists \ \nu : Y \to \mathcal{M}(\widehat{X})$ so that

$$G(f)(x) = \int_X f d\nu_x \ \forall \ f \in C_{\mathtt{lim}}(X),$$

Next, fix $f \in C_{\mathtt{lim}}(X)$ & $f_n \in \mathtt{Span}\, J$ with $\|f_n - f\|_{C_{\mathtt{lim}}(X)} =: \varepsilon_n \xrightarrow[n \to \infty]{} 0$. This entails $\|G(f_n) - G(f)\|_{C(Y)} \le C\varepsilon_n$, whence, $\forall \ k, n \ge 1$,

$$\begin{aligned} |\mu_{k,x}(f) - \int_{\widehat{X}} f d\nu_x| &= |\mu_{k,x}(f) - G(f)(x)| \\ &\le |\mu_{k,x}(f_n) - G(f_n)(x)| + |\mu_{k,x}(f) - \mu_{k,x}(f_n)| + |G(f)(x) - G(f_n)(x)| \\ &\le |\mu_{k,x}(f_n) - G(f_n)| + 2C\varepsilon_n. \end{aligned}$$

Given $\mathcal{E} > 0$, fix $N = N_{\mathcal{E}} \ge 1$ so that $\varepsilon_N < \frac{\mathcal{E}}{4C}$ and then choose $K = K_{\mathcal{E}} \ge 1$ so that

$$|\mu_{k,x}(f_N) - G(f_N)| < \tfrac{\mathcal{E}}{2C} \ \forall \ x \in Y, \ k \ge K$$

entailing

$$|\mu_{k,x}(f) - G(f)(x)| \le \mathcal{E} \ \forall \ x \in Y, \ k \ge K.$$

Thus $\nu \in C(Y, \mathcal{M}(\widehat{X}))$ and $\mu_n \xrightarrow[n \to \infty]{C(Y, \mathcal{M}(\widehat{X}))} \nu$. ☑

For $Y \subset \mathbb{R}$ a bounded interval, define the *Lipschitz norm* of $F : Y \to \mathbb{R}$ by

$$\|F\|_{\mathtt{Lip(Y)}} := \|F\|_{L^\infty(Y,m)} + \mathtt{Lip}_Y(F)$$

where

$$\mathtt{Lip}_Y(F) := \sup_{x,y \in Y,\ x \neq y} \tfrac{|F(x)-F(y)|}{|x-y|}.$$

Note that if $\mathtt{Lip}_Y(F) < \infty$, then F is absolutely continuous on Y and $\mathtt{Lip}_Y(F) = \|F'\|_{L^\infty(Y,m)}$.

Lemma 2.1.

Let $T : \mathbb{R}^{2+} \circlearrowleft$ be a normalised, parabolic inner function with Pommerenke limit $\Pi = U + iV$.

Suppose that either (i) T is exact or (ii) $T^n(i) \xrightarrow[n\to\infty]{} \infty$ tangentially and let $Y \subset \mathbb{R}$ be a bounded interval, then $\forall\ z \in \mathbb{R}^{2+}$,

☞ $$F_{n,z} := \tfrac{1}{u_n}\widehat{T}^n\varphi_z \xrightarrow[n\to\infty]{} V(z) \text{ uniformly on } Y \text{ with } u_n := \widehat{T}^n\varphi_i(0);$$

👍 $$\sup_{n\geq 1} \|F_{n,z}\|_{\mathtt{Lip(Y)}} < \infty$$

Proof of ☞

Write $T^n(z) = \alpha_n(z) + i\beta_n(z)$ & $T^n(i) = a_n + ib_n$. By Pommerenke's theorem 1.3

$$\tfrac{\alpha_n(z)-a_n}{b_n} \xrightarrow[n\to\infty]{} U(z),\ \ \tfrac{\beta_n(z)}{b_n} \xrightarrow[n\to\infty]{} V(z).$$

It follows that $\beta_n(z) \sim b_nV(z)$ and that

☆ $$\begin{aligned}\alpha_n(z)^2 + \beta_n(z)^2 &= (a_n + U(z) + o(b_n))^2 + V(z)^2b_n^2 + o(b_n^2)\\ &= (a_n + U(z))^2 + V(z)^2b_n^2 + 2a_no(b_n) + o(b_n^2)\\ &= a_n^2 + V(z)^2b_n^2 + o(a_n^2 + b_n^2)\end{aligned}$$

Thus,

✈ $$\begin{aligned}F_{n,z}(x) &= \tfrac{1}{u_n}\widehat{T}^n\varphi_z(x) = \tfrac{\beta_n(z)}{b_n}\cdot\tfrac{a_n^2+b_n^2}{(\alpha_n(z)-x)^2+\beta_n(z)^2}\\ &\sim V(z)\tfrac{a_n^2+b_n^2}{a_n^2+V(z)^2b_n^2} \quad \text{uniformly on } Y\\ &\sim \tfrac{V(z)(\Delta_n+1)}{\Delta_n+V(z)^2} \quad \text{with} \quad \Delta_n := (\tfrac{a_n}{b_n})^2.\end{aligned}$$

In case (i), $\Delta_n \to 0,\ U \equiv 0$ and $V \equiv 1$ entailing

$$\tfrac{1}{u_n}\widehat{T}^n\varphi_z \xrightarrow[n\to\infty]{\mathtt{uniformly\ on}\ Y} 1 = V(z).$$

In case (ii), $\Delta_n \to \infty$ entailing

$$\tfrac{1}{u_n}\widehat{T}^n\varphi_z \xrightarrow[n\to\infty]{\mathtt{uniformly\ on}\ Y} V(z).\ \ ☑ ☞$$

Proof of 👍

By ☞, $\sup_{n\geq 1}\|F_{n,z}\|_{C_{\lim}(X)} < \infty\ \forall\ z \in \mathbb{R}^{2+}$.

Thus, for $\sup_{n\geq 1}\|F_{n,z}\|_{\mathtt{Lip(Y)}} < \infty$, it suffices that each $F_{n,z}$ is absolutely continuous with

⛴ $$\sup_{n\geq 1}\|F_{n,z}'\|_{L^\infty(Y,m)} < \infty.$$

Uniformly in $x \in Y$:

$$\begin{aligned}|F'_{n,z}(x)| &= \tfrac{\beta_n(z)}{\pi u_n}\left|\tfrac{d}{dx}\left(\tfrac{1}{(\alpha_n(z)-x)^2+\beta_n(z)^2}\right)\right| \\ &= \tfrac{\beta_n(z)}{\pi u_n}\tfrac{2|\alpha_n(z)-x|}{((\alpha_n(z)-x)^2+\beta_n(z)^2)^2} \\ &= V(z)\tfrac{2(a_n^2+b_n^2)|\alpha_n(z)-x|}{((\alpha_n(z)-x)^2+\beta_n(z)^2)^2} \\ &\lesssim V(z)\tfrac{|\alpha_n(z)|+M}{a_n^2+b_n^2} \ \text{ where } Y \subset \ [-M,M] \\ &\xrightarrow[n\to\infty]{} 0. \quad ☑ \quad ⛴\end{aligned}$$

§2.2 Proof of Theorem M.

Proof that dual mixing ⇒ exactness

Write $T^n(z) = \alpha_n(z) + i\beta_n(z)$ & $T^n(i) = a_n + ib_n$. By Pommerenke's theorem 1.3

$$\tfrac{\alpha_n(z)-a_n}{b_n} \xrightarrow[n\to\infty]{} U(z), \ \ \tfrac{\beta_n(z)}{b_n} \xrightarrow[n\to\infty]{} V(z)$$

where $\Pi = U + iV \in \mathbb{R}^{2+}$ is the Pommerenke limit and in particular $V > 0$.

For $z \in \mathbb{R}^{2+}$, as $n \to \infty$:

$$\begin{aligned}a_n^2 + b_n^2 &= \tfrac{b_n}{\pi\widehat{T}^n\varphi_i(0)} \sim \tfrac{b_n}{\pi\widehat{T}^n\varphi_z(0)} \ \text{by dual mixing} \\ &\sim \tfrac{1}{V(z)}\tfrac{\beta_n(z)}{\pi\widehat{T}^n\varphi_z(0)} = \tfrac{\alpha_n(z)^2+\beta_n(z)^2}{V(z)} \\ &\sim \tfrac{a_n^2+V(z)^2b_n^2}{V(z)} \ \text{by } ☆.\end{aligned}$$

Thus $\forall\ z \in \mathbb{R}^{2+},\ \exists\ \varepsilon_n = \varepsilon_n(z) \downarrow 0$ so that

$$V(z)\Delta_n + V(z) = (1 \pm \varepsilon_n)(\Delta_n + V(z)^2).$$

- If $\exists\ n_k \to \infty$ with $\Delta_{n_k} \xrightarrow[k\to\infty]{} \infty$, then $V \equiv 1$.
- If $\exists\ n_k \to \infty,\ \Delta_{n_k} \xrightarrow[k\to\infty]{} r \in [0,\infty)$, then $\forall\ z \in \mathbb{R}^{2+}$,

$$\begin{aligned}V(z)r + V(z) &\xleftarrow[k\to\infty]{} V(z)\Delta_{n_k} + V(z) \\ &= (1 \pm \varepsilon_{n_k})(\Delta_{n_k} + V(z)^2) \\ &\xrightarrow[k\to\infty]{} r + V(z)^2\end{aligned}$$

and $V(z) \in \{1, r\}$.

Thus V being continuous, is constant, whence also the Pommerenke limit and $(\mathbb{R}, m, T)$ is exact. ☑

Proof that exactness ⇒ dual mixing

We show that for $f \in C_C(\mathbb{R})$,

🖉 $$\mu_{n,x}\left(\tfrac{f}{\varphi_i}\right) = \tfrac{1}{u_n}\widehat{T}^n f \xrightarrow[n\to\infty]{\text{u.c.s.}} \int_{\mathbb{R}} f\,dm$$

with $u_n := \widehat{T}^n\varphi_i(0)$ and $\mu_n : \mathbb{R} \to \mathcal{M}(\mathbb{R})$ defined by

$$\mu_{n,x}(f) := \tfrac{\widehat{T}^n(f\varphi_i)(x)}{u_n}.$$

By Lemma 2.1 ☞ (p. 8), 🖉 holds for $f = \varphi_z,\ z \in \mathbb{R}^{2+}$.

Now fix a bounded subinterval Y of $\mathbb{R}$. By 👍, for any $z \in \mathbb{R}^{2+}$,

$$\sup_{n\ge 1} \|F_{n,z}\|_{\mathtt{Lip}(Y)} < \infty$$

where $F_{n,z}(x) := \mu_{n,x}(\frac{\varphi_z}{\varphi_i}) = \frac{\widehat{T}^n \varphi_z(x)}{u_n}$.

Thus $\mathcal{K} := \{\mu_n|_Y : \ n \ge 1\}$ satisfies 🍃 (p. 7) with respect to $\operatorname{Span} J$ for $J = \{\varphi_z : \ z \in \mathbb{R}^{2+}\} \cup \{\mathbb{1}\}$ and by Proposition 2.1 is precompact in $C(Y, \mathcal{M}(\widehat{\mathbb{R}}))$.

Next, suppose that $\nu \in \mathcal{K}'$, i.e. $\exists\ n_k \to \infty$ & $\nu \in \mathcal{M}(\widehat{\mathbb{R}})$ are so that $\mu_{n_k} \xrightarrow[k\to\infty]{\mathcal{M}(\widehat{\mathbb{R}})} \nu$.
By ☞,

$$\mu_{n_k,x}(\tfrac{\varphi_z}{\varphi_i}) \xrightarrow[k\to\infty]{} 1 = P_i(\tfrac{\varphi_z}{\varphi_i}) \ \forall\ z \in \mathbb{R}^{2+}$$

with the conclusion that $\nu_x = P_i\ \forall\ x \in Y$ and $\mu_{n_k} \xrightarrow[k\to\infty]{C(Y,\mathcal{M}(\mathbb{R}))} P_i$.

This shows 💉 $\forall\ f \in C_{\mathtt{lim}}(\mathbb{R})$ and completes the proof of Theorem M. ☑

§2.3 Proof of Theorem Q.

Write $T^n(z) = \alpha_n(z) + i\beta_n(z)$ & $T^n(i) = a_n + ib_n$. By Pommerenke's theorem 1.3

$$\tfrac{\alpha_n(z)-a_n}{b_n} \xrightarrow[n\to\infty]{} U(z), \ \tfrac{\beta_n(z)}{b_n} \xrightarrow[n\to\infty]{} V(z)$$

where $U + iV \in \mathbb{R}^{2+}$ is the Pommerenke limit and in particular $V > 0$.

We'll show that for $f \in C_C(\mathbb{R})$,

$$\text{⛑} \qquad \tfrac{1}{u_n}\widehat{T}^n f \xrightarrow[n\to\infty]{\text{u.c.s.}} \int_{\mathbb{R}} f\,d\rho$$

with $u_n := \widehat{T}^n\varphi_i(0)$ and $d\rho(x) := (1+x^2)d\varpi_\Pi(x)$.

Here, as before, $\varpi_\Pi \in \mathcal{P}(\widehat{\mathbb{R}})$ is the representing measure for Π as in 🔒. Since T is not Möbius, $\varpi_\Pi \ne \delta_\infty$ and $\rho \not\equiv 0$.

By ☞ (on p. 8), ⛑ holds for $f = \varphi_z,\ z \in \mathbb{R}^{2+}$.

As before, define $\mu_n : \mathbb{R} \to \mathcal{M}(\mathbb{R})$ by

$$\mu_{n,x}(f) := \tfrac{\widehat{T}^n(f\varphi_i)(x)}{u_n}$$

and

$$F_{n,z}(x) := \mu_{n,x}(\tfrac{\varphi_z}{\varphi_i}) = \tfrac{\widehat{T}^n\varphi_z(x)}{u_n}.$$

By 👍 (on p. 8), for any $z \in \mathbb{R}^{2+}$ & bounded subinterval Y of $\mathbb{R}$,

$$\sup_{n\ge 1} \|F_{n,z}\|_{\mathtt{Lip}(Y)} < \infty.$$

It follows that $\mathcal{K} := \{\mu_n|_Y : \ n \ge 1\}$ satisfies 🍃 (on p. 7) with respect to $\operatorname{Span} J$ for $J = \{\varphi_z : \ z \in \mathbb{R}^{2+}\} \cup \{\mathbb{1}\}$.

Thus by Proposition 2.1, $\mathcal{K}$ is precompact in $C(Y, \mathcal{M}(\widehat{\mathbb{R}}))$ and

$$\mu_n \xrightarrow[n\to\infty]{C(Y,\mathcal{M}(\widehat{\mathbb{R}}))} \varpi_\Pi.$$

In particular for $f \in C_C(\mathbb{R})$,

$$\mu_n(f) \xrightarrow[n\to\infty]{} \varpi_\Pi(f) \text{ uniformly on } Y\ \forall\ f \in C_C(\mathbb{R})$$

and

$$\tfrac{1}{u_n}\widehat{T}^n f = \mu_n\big(\tfrac{f}{\varphi_i}\big) \xrightarrow[n\to\infty]{\text{u.c.s.}} \varpi_\Pi\big(\tfrac{f}{\varphi_i i}\big) = \int_{\mathbb{R}} f(t)d\rho(t).$$

This proves Theorem Q. ☑

§3 Conditions for mixing & quasi-mixing

Proposition 3.1. *Let $T : \mathbb{R}^{2+} \circlearrowleft$ be a non-Möbius, normalised, parabolic inner function with representation*

$$T(z) = z + \beta + \int_{\mathbb{R}} \tfrac{d\nu(t)}{t-z} \ \textit{with}\ \nu \in \mathcal{M}(\mathbb{R}),\ \nu \perp m\ \&\ \beta \in \mathbb{R}.$$

(i) *If $\beta = 0$, then $(\mathbb{R}, m, T)$ is conservative and dual mixing with $u_n(T) = \frac{1}{\pi\sqrt{2\nu(\mathbb{R})n}}$.*

(ii) *If $\beta \neq 0$, then $(\mathbb{R}, m, T)$ is singular dual mixing: satisfying 🕮 with respect to $d\rho_\Pi(x) := \frac{d\varpi_\Pi(x)}{\varphi_i(x)}$; with $u_n(T) = \frac{b_\infty}{\pi(n\beta)^2}$ where $b_\infty := \lim_{n\to\infty} \mathtt{Im}T^n(i) < \infty$ and ϖ_Π is the representing measure of Π as in 🔒.*

By [Ham94, Theorem P], ρ in (ii) is singular and $\rho \not\equiv 0$ because T is not Möbius whence $\varpi_\Pi \neq \delta_\infty$.

Proof of (i) Set $T^n(i) := a_n + ib_n$, then $b_n \sim \sqrt{2\nu(\mathbb{R})n}$ & $a_n = o(b_n)$ as $n \to \infty$. This is established in [Aar97, Theorem 6.4.1] for ν compactly supported and may be extracted from [ASW21, §4] for general ν.

Thus $(\mathbb{R}, m, T)$ is conservative, hence exact (see [Aar97, Chapter 6]); and thus dual mixing by Theorem M. ☑ (i)

Proof of (ii) Here $a_n = \beta n + o(n)$ and $b_n \uparrow b_\infty < \infty$. See [Aar97, Theorem 6.4.1] for ν compactly supported and [Ivr] for general ν.

Thus $T^n(i) \to \infty$ tangentially and ρ_Π-quasi mixing is established by Theorem Q with $\widehat{T}^n\varphi_i(0) \sim \frac{b_\infty}{\pi(n\beta)^2} =: u_n(T)$. ☑ (ii)

References

[Aar78] Jon Aaronson. Ergodic theory for inner functions of the upper half plane. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 14(3):233–253, 1978.

[Aar81] Jon Aaronson. A remark on the exactness of inner functions. *J. London Math. Soc. (2)*, 23(3):469–474, 1981.

[Aar97] Jon Aaronson. *An introduction to infinite ergodic theory*, volume 50 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[Aar13] Jon Aaronson. Rational weak mixing in infinite measure spaces. *Ergodic Theory Dynam. Systems*, 33(6):1611–1643, 2013.

[ASW21] Octavio Arizmendi, Mauricio Salazar, and Jiun-Chau Wang. Berry-Esseen type estimate and return sequence for parabolic iteration in the upper half-plane. *Int. Math. Res. Not. IMRN*, (23):18037–18056, 2021.

[AT20] Jon Aaronson and Dalia Terhesiu. Local limit theorems for suspended semiflows. *Discrete Contin. Dyn. Syst.*, 40(12):6575–6609, 2020.

[AW73] Roy L. Adler and Benjamin Weiss. The ergodic infinite measure preserving transformation of Boole. *Israel J. Math.*, 16:263–278, 1973.

[Boo57] George Boole. On the comparison of transcendents, with certain applications to the theory of definite integrals. *Philosophical Transactions of the Royal Society of London*, 147:745–803, 1857.

[DN22] Dmitry Dolgopyat and Péter Nándori. Infinite measure mixing for some mechanical systems. *Adv. Math.*, 410(part B):Paper No. 108757, 56, 2022.

[FR24] Livio Flaminio and Davide Ravotti. Abelian covers of hyperbolic surfaces: equidistribution of spectra and infinite volume mixing asymptotics for horocycle flows. *Nonlinearity*, 37(7):Paper No. 075015, 26, 2024.

[Gla77] J. W. L. Glaisher. Note on certain theorems in definite integration. Messenger (2) VII. 63-74 (1877)., 1877.

[Ham94] D. H. Hamilton. Absolutely continuous conjugacies of Blaschke products. III. *J. Anal. Math.*, 63:333–348, 1994.

[Hop37] E. Hopf. *Ergodentheorie*. Number v. 5, no. 2 in Ergebnisse der Mathematik und ihrer Grenzgebiete, 5. Bd. Julius Springer, 1937.

[Ivr] Oleg Ivrii. Dynamics of simply parabolic inner functions. *arXiv:2606.19260*.

[Kri67] Klaus Krickeberg. Strong mixing properties of markov chains with infinite invariant measure. In *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, Volume 2: Contributions to Probability Theory, Part 2*, pages 431–446, Berkeley, Calif., 1967. University of California Press.

[Kri69] K. Krickeberg. Recent results on mixing in topological measure spaces. In *Probability and Information Theory (Proc. Internat. Sympos., McMaster Univ., Hamilton, Ont., 1968)*, pages 178–185. Springer, Berlin, 1969.

[Len10] Marco Lenci. On infinite-volume mixing. *Comm. Math. Phys.*, 298(2):485–514, 2010.

[Let77] Gérard Letac. Which functions preserve Cauchy laws? *Proc. Amer. Math. Soc.*, 67(2):277–286, 1977.

[Lin71] Michael Lin. Mixing for Markov operators. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 19:231–242, 1971.

[MT12] Ian Melbourne and Dalia Terhesiu. Operator renewal theory and mixing rates for dynamical systems with infinite measure. *Invent. Math.*, 189(1):61–110, 2012.

[Neu78] J. H. Neuwirth. Ergodicity of some mappings of the circle and the line. *Israel J. Math.*, 31(3-4):359–367, 1978.

[Nor68] Eric A. Nordgren. Composition operators. *Canadian J. Math.*, 20:442–449, 1968.

[Ore61] Steven Orey. Strong ratio limit property. *Bull. Amer. Math. Soc.*, 67:571–574, 1961.

[Ore71] Steven Orey. *Limit theorems for Markov chain transition probabilities*. Van Nostrand Reinhold Mathematical Studies, No. 34. Van Nostrand Reinhold Co., London-New York-Toronto, 1971.

[Pap67] F. Papangelou. Strong ratio limits, R-recurrence and mixing properties of discrete parameter Markov processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 8:259–297, 1967.

[Pom79] Ch. Pommerenke. On the iteration of analytic functions in a halfplane. *J. London Math. Soc. (2)*, 19(3):439–447, 1979.

[Pru65] William E. Pruitt. Strong ratio limit property for R-recurrent Markov chains. *Proc. Amer. Math. Soc.*, 16:196–200, 1965.

[PT25] Françoise Pène and Dalia Terhesiu. Strong mixing for the periodic Lorentz gas flow with infinite horizon. *Trans. Amer. Math. Soc.*, 378(3):1619–1679, 2025.

[Tha00] Maximilian Thaler. The asymptotics of the Perron-Frobenius operator of a class of interval maps preserving infinite measures. *Studia Math.*, 143(2):103–119, 2000.

School of Math. Sciences, Tel Aviv University, 69978 Tel Aviv, Israel.
Email address: aaro@tau.ac.il